The material of this section has been studied and written by Ha Tu Bui.

Theorem *Let G be a simple undirected finite graph:*

(i) G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G)+1$ (provided $\delta(G) \geq 2$).

(ii) If G contains a cycle, then $\operatorname{girth}(G) \leq 2\operatorname{diam}(G)+1$.

(iii) If $\operatorname{rad}(G)=k$ and $\Delta(G)=d \geq 3$, then G contains at most $\frac{d}{d-2}(d-1)^k$ vertices.

Proof of Theorem Observe that since the graph G is simple and undirected, a path is uniquely defined by the enumeration of the successive vertices. In this proof, the description of the paths will be done accordingly.

(i) Let $\delta(G)=k$. Start at any vertex v_0. If $k \geq 1$ then v_0 is adjacent to some vertex v_1. The path (v_0, v_1) has length 1. If $k \geq 2$ then v_1 is adjacent to a vertex $v_2 \neq v_0$. The path (v_0, v_1, v_2) has length 2. If $k \geq 3$ then v_2 is adjacent to a vertex $v_3 \notin \{v_0, v_1\}$. The path (v_0, v_1, v_2, v_3) has length 3. We repeat this argument until we have chosen v_k adjacent to v_{k-1} and $v_k \notin \{v_0, v_1, ..., v_{k-2}\}$. As a result, G contains the path $(v_0, v_1, v_2, ..., v_k)$, whose length is $k=\delta(G)$.

Assume that the longest path in G is $M=(v_0, v_1, ..., v_{m-1}, v_m)$. Note that v_m cannot be adjacent to a vertex $x \notin \{v_0, v_1, ..., v_{m-1}\}$ because it would produce a path longer than M, a contradiction. Then we have v_m adjacent to at least k vertices belonging to the set $\{v_0, v_1, ..., v_{m-1}\}$. Consequently, at least one of these k vertices is in the set $H=\{v_0, v_1, ..., v_{m-k}\}$. When v_m is adjacent to one vertex in H, it produces a cycle of length $\geq k+1$ (The equality holds for the case v_m and v_{m-k} are adjacent). Thus, if $\delta(G) \geq 2$ then G contains a cycle of length at least $\delta(G)+1$.

(ii) Let C be the shortest cycle in G. By definition, the length of C is $\operatorname{girth}(G)$. Assume that $\operatorname{girth}(G) \geq 2\operatorname{diam}(G)+2$. Then, there exists two vertices x and y in C such that the two paths between x and y in C are the path of length $\operatorname{diam}(G)+1$, called P_C, and the other path whose length is at least $\operatorname{diam}(G)+1$. The distance between x and y in C is $\mathrm{d}_C(x,y)=\operatorname{diam}(G)+1$. Let $\mathrm{d}_G(x,y)$ be the distance between x and y in G and it corresponds to the shortest path P_G in G between these two vertices. Observe that $\mathrm{d}_G(x,y) \leq \operatorname{diam}(G)$ (based on the definition of diameter) while $\mathrm{d}_C(x,y)=\operatorname{diam}(G)+1$. As a result, P_G is not a subgraph of C. P_G and P_C have two independent vertices in common, namely z_1 and z_2, which satisfy the following condition: The path contained in P_G between z_1 and z_2 and the path contained in P_C between z_1 and z_2 do not have any vertex in common, except z_1 and z_2. In some graphs, it is possible to choose $z_1 \equiv x$ or $z_2 \equiv y$. However, it is not always true for any graph. For example, the graph shown in Figure 1.20 has z_1 and z_2 different from x and y.

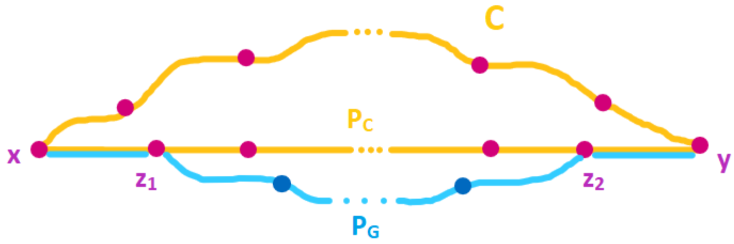

Fig. 1.20. A shorter cycle

The composition of the path contained in P_G between z_1 and z_2 and the path contained in P_C between z_1 and z_2 forms a cycle of length: $L \leq \mathrm{d}_G(x,y) + \mathrm{d}_C(x,y) = \mathrm{diam}(G) + \big((\mathrm{diam}(G)+1\big) = 2\mathrm{diam}(G)+1$. Thus, this cycle is even shorter than C, which is a contradiction. Therefore, the assumption is not correct and we have proved $\mathrm{girth}(G) \leq 2\mathrm{diam}(G)+1$. One example that the equality holds: G is a cycle having $(2n+1)$ vertices, then $\mathrm{diam}(G) = n$ and $\mathrm{girth}(G) = 2n+1 = 2\mathrm{diam}(G)+1$.

(iii) Let c be a central vertex of G. Let V_i be the set of vertices of G at distance i from c, with $i \in \{0,1,...,k\}$. Then $V(G) = \cup_0^k V_i$. By induction, we are going to prove that for any $i \in \{0,1,...,k\}$,

$$|V_{i+1}| \leq d(d-1)^i \tag{1.3}$$

It is clear that $|V_1| \leq d \Rightarrow |V_1| \leq d(d-1)^0$, which means that (1.3) is correct for $i=0$. For $i \in \{1,...,k-1\}$, we are going to show that if $|V_i| \leq d(d-1)^{i-1}$ is true then $|V_{i+1}| \leq d(d-1)^i$ is true. Indeed, observe that for $i \geq 1$, each vertex in V_i is connected to at most $(d-1)$ vertices in V_{i+1}, see Figure 1.21. As a consequence, one has

$$|V_{i+1}| \leq (d-1)\,|V_i| \leq (d-1)d(d-1)^{i-1} = d(d-1)^i.$$

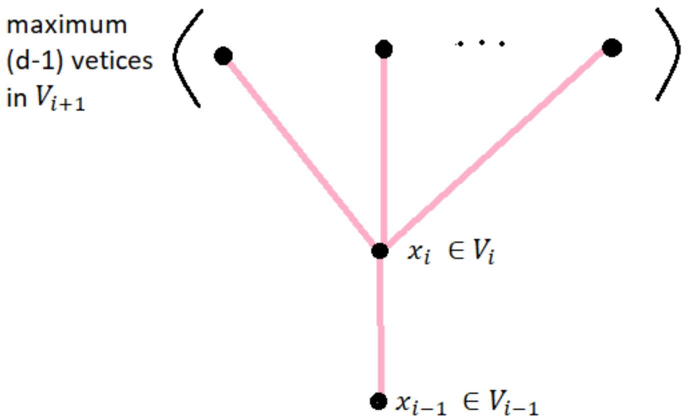

Fig. 1.21. Maximal number of adjacent vertices in V_{i+1}

Thus, for the number of vertices contained in G one has when $d \geq 3$:

$$
\begin{aligned}
|V(G)| &= \left|\cup_0^k V_i\right| = 1 + \sum_{i=1}^{k} |V_i| \leq 1 + \sum_{i=0}^{k-1} d(d-1)^i = 1 + d\sum_{i=0}^{k-1}(d-1)^i \\
&= 1 + d\frac{(d-1)^k - 1}{(d-1) - 1} = 1 + \frac{d}{d-2}\big((d-1)^k - 1\big) = \frac{d}{d-2}(d-1)^k - \frac{2}{d-2} \\
&< \frac{d}{d-2}(d-1)^k.
\end{aligned}
$$

Therefore, G contains at most $\frac{d}{d-2}(d-1)^k$ vertices. □