Good luck

Exercise 1 *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *given by* $f(x,y) = x^2y + 3x^4y^3 - 4$, *and consider the curve* L_0 *in* $\mathbb{R}^2$ *defined by* $L_0 = \big\{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\big\}$. *Check that* $(1,1)$ *belongs to* L_0, *and determine the equation of the tangent of* L_0 *at this point.*

Exercise 2 *Compute the integral* $\iint_\Omega x \,\mathrm{d}x\,\mathrm{d}y$ *with* Ω *the subset of* $\mathbb{R}^2$ *defined by the three lines of equation* $x = 0$, $y = x + 2$, *and* $y = -x$.

Exercise 3 *Find the mass of a spherical ball of radius* R *if the density of the ball is given in spherical coordinates by the function* $\rho(r,\theta,\varphi) = r\sin\left(\frac{\theta}{2}\right)$.

Exercise 4 *Check the validity of Green's theorem for the function* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *defined for* $(x,y) \in \mathbb{R}^2$ *by* $f(x,y) = \begin{pmatrix} y^2 \\ x \end{pmatrix}$ *on the circle of radius* 2 *and centered at* $(0,0)$ *in* $\mathbb{R}^2$. *In other terms, by two different approaches you should get the same result.*

Exercise 5 *The aim of this exercise is to show that the area of a surface is independent of its parametrization. Consider an open set* $\Omega \subset \mathbb{R}^2$, *a function* $g : \Omega \to \mathbb{R}$ *of class* C^1, *and the parametric surface given by*

$$\Omega \ni (s,t) \mapsto q(s,t) = \begin{pmatrix} s \\ t \\ g(s,t) \end{pmatrix} \in \mathbb{R}^3$$

Let also $\varphi : \Lambda \ni (x,y) \mapsto \big(\varphi_1(x,y), \varphi_2(x,y)\big) \in \Omega$ *be a diffeomorphism of class* C^1.

(i) Write the formula for the computation of the area of $q(\Omega)$,

(ii) Compute $[\partial_1(g\circ\varphi)](x,y)$ *and* $[\partial_2(g\circ\varphi)](x,y)$ *(altogether you should obtain 4 terms),*

(iii) Using the previous information, compute the vectors $[\partial_1(q\circ\varphi)](x,y)$ *and* $[\partial_2(q\circ\varphi)](x,y)$,

(iv) Compute $\Big([\partial_1(q\circ\varphi)](x,y)\Big) \times \Big([\partial_2(q\circ\varphi)](x,y)\Big)$, *and factor the common term* $\Big[(\partial_1\varphi_1)(\partial_2\varphi_2) - (\partial_1\varphi_2)(\partial_2\varphi_1)\Big](x,y)$,

(v) Conclude with the general formula for the change of variables for a volume integral that the area of a surface is independent of its parametrization, or more precisely show that the area of $q(\Omega)$ *is equal to the area of* $[q\circ\varphi](\Lambda)$.

Exam Calculus II 2020

Total : 20 pts

Exercise 1 4 pts

- $f(1,1) = 1^2 1 + 3\, 1^4 1^3 - 4 = 0 \Rightarrow (1,1) \in L_0.$ 1

- $\partial_y f(x,y) = x^2 + 9x^4 y^2$

$\partial_y f(1,1) = 1 + 9 = 10 \neq 0$ the implicit function thm can be applied. Consider $\phi : (1-\varepsilon, 1+\varepsilon) \to \mathbb{R}$ with $\phi(1) = 1$ and such that $f(x, \phi(x)) = 0$ $\forall x \in (1-\varepsilon, 1+\varepsilon)$.

Then $\frac{d}{dx} f(x, \phi(x)) = 2x\phi(x) + x^2 \phi'(x) + 12x^3 \phi(x)^3$

$$+ 9x^4 \phi^2(x)\, \phi'(x) = 0$$ 1

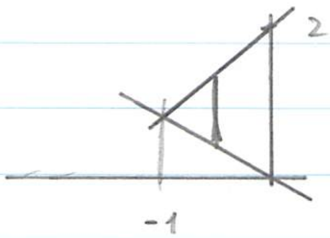
At $x = 1$: $2 + \phi'(1) + 12 + 9\phi'(1) = 0$

$$\Leftrightarrow \phi'(1) = -\frac{14}{10} = -\frac{7}{5}.$$

Tangent line : $(y-1) = -\frac{7}{5}(x-1)$

$$\Leftrightarrow y = -\frac{7}{5}x + \frac{12}{5}.$$ 2

Exercise 2 — 3 pts



$$\iint_\Omega x \, dx \, dy$$

$$= \int_{-1}^{0} x \int_{-x}^{x+2} 1 \, dy$$

$$= \int_{-1}^{0} x \left(x + 2 - (-x) \right) dx$$

$$= \int_{-1}^{0} x \, (2x + 2) \, dx$$

$$= 2 \int_{-1}^{0} (x^2 + x) \, dx$$

$$= \frac{2}{3} x^3 \Big|_{-1}^{0} + x^2 \Big|_{-1}^{0}$$

$$= \frac{2}{3} - 1 = \underline{\underline{-\frac{1}{3}}} \, .$$

Exercise 3 — 3 pts

$$\iiint_B \rho(r, \theta, \varphi) \, r^2 \sin(\varphi) \, dr \, d\theta \, d\varphi$$

$$= \int_0^R r^3 \, dr \int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta \int_0^{\pi} \sin(\varphi) \, d\varphi$$

$$= \frac{1}{4} R^4 \; 2 \, (-) \cos\left(\frac{\theta}{2}\right) \Big|_0^{2\pi} \; (-) \cos(\varphi) \Big|_0^{\pi}$$

$$= \frac{1}{2} R^4 \; 2 \cdot 2 = \underline{2 R^4} \, .$$

Exercise 4   5 pts

1) Parametrization circle : $\theta \mapsto (2\cos(\theta), 2\sin(\theta))$ 1

$$\int_c J = \int_0^{2\pi} J(2\cos(\theta), 2\sin(\theta)) \cdot 2\begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} d\theta$$

$$= 2\int_0^{2\pi} \begin{pmatrix} 4\sin(\theta)^2 \\ 2\cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} d\theta$$

$$= 2\int_0^{2\pi} \left(-4\sin^3(\theta) + 2\cos^2(\theta)\right) d\theta$$

$$= 2\left(0 + 2\,\frac{2\pi}{2}\right) = \underline{\underline{4\pi}}.$$ 2

Same result

2) $$\iint_D (1-2y)\,dx\,dy = \iint_D 1\,dx\,dy$$

$$= \pi\, 2^2 = \underline{\underline{4\pi}}.$$ 2

Some explanations should be provided.

Remark : We have used that

- $\int_0^{2\pi} \sin^3(\theta)\,d\theta = 0$ (by symmetry !)
- $\int_0^{2\pi} \cos^2(\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi} \left(\cos^2(\theta) + \sin^2(\theta)\right) d\theta$

$$= \frac{1}{2}\, 2\pi = \pi$$

- $\iint_D y\,dy\,dx = 0$ (by symmetry)

Exercise 5

5 pts

i) $$\text{Area} = \iint_{\Omega} \| \partial_1 q(s,t) \times \partial_2 q(s,t) \| \, ds \, dt \, .$$

ii) $$\partial_j g(\varphi_1(x,y), \varphi_2(x,y)) =$$

$$= \partial_1 g(\varphi(x,y)) \, \partial_j \varphi_1(x,y) + \partial_2 g(\varphi(x,y)) \, \partial_j \varphi_2(x,y)$$

for $j = 1, 2$.

iii) $$q(\varphi(x,y)) = \begin{pmatrix} \varphi_1(x,y) \\ \varphi_2(x,y) \\ g(\varphi(x,y)) \end{pmatrix}$$

$$\Rightarrow \partial_1 \big(q(\varphi(x,y)) \big) = \begin{pmatrix} \partial_1 \varphi_1(x,y) \\ \partial_1 \varphi_2(x,y) \\ \partial_1 (g(\varphi(x,y))) \end{pmatrix}$$

$$\partial_2 \big(q(\varphi(x,y)) \big) = \begin{pmatrix} \partial_2 \varphi_1(x,y) \\ \partial_2 \varphi_2(x,y) \\ \partial_2 (g(\varphi(x,y))) \end{pmatrix}$$

Computed there

iv) $$\partial_1 \big(q(\varphi(x,y)) \big) \times \partial_2 \big(q(\varphi(x,y)) \big) =$$

$$= \begin{pmatrix} \partial_1 \varphi_2(x,y) \big([\partial_1 g](\varphi(x,y)) \, \partial_2 \varphi_1(x,y) + [\partial_2 g](\varphi(x,y)) \, \partial_2 \varphi_2(x,y) \big) \\ \quad - \partial_2 \varphi_2(x,y) \big([\partial_1 g](\varphi(x,y)) \, \partial_1 \varphi_1(x,y) + [\partial_2 g](\varphi(x,y)) \, \partial_1 \varphi_2(x,y) \big) \\ \\ - \partial_1 \varphi_1(x,y) \big([\partial_1 g](\varphi(x,y)) \, \partial_2 \varphi_1(x,y) + [\partial_2 g](\varphi(x,y)) \, \partial_2 \varphi_2(x,y) \big) \\ \quad + \partial_2 \varphi_1(x,y) \big([\partial_1 g](\varphi(x,y)) \, \partial_1 \varphi_1(x,y) + [\partial_2 g](\varphi(x,y)) \, \partial_1 \varphi_2(x,y) \big) \\ \\ \partial_1 \varphi_1(x,y) \, \partial_2 \varphi_2(x,y) - \partial_2 \varphi_1(x,y) \, \partial_1 \varphi_2(x,y) \end{pmatrix}$$

$$= \begin{pmatrix} \partial_1\varphi_1(x,y)\ \partial_2\varphi_2(x,y) - \partial_1\varphi_2(x,y)\ \partial_2\varphi_1(x,y) \end{pmatrix} \begin{pmatrix} -[\partial_1 g](\varphi(x,y)) \\ -[\partial_2 g](\varphi(x,y)) \\ 1 \end{pmatrix}$$

$$= \det J_\varphi(x,y) \begin{pmatrix} -[\partial_1 g](\varphi(x,y)) \\ -[\partial_2 g](\varphi(x,y)) \\ 1 \end{pmatrix}.$$

v) $\underline{\text{Area } g(\varphi(\Lambda))}$ = by def

$$= \iint_\Lambda \| \partial_1 (g(\varphi(x,y))) \wedge \partial_2 (g(\varphi(x,y))) \| \, dx\, dy$$

point iv)

$$= \iint_\Lambda \left\| \det J_\varphi(x,y) \begin{pmatrix} -[\partial_1 g](\varphi(x,y)) \\ -[\partial_2 g](\varphi(x,y)) \\ 1 \end{pmatrix} \right\| \, dx\, dy$$

the absolute value is not necessary

$$= \iint_\Lambda \left\| \begin{pmatrix} 1 \\ 0 \\ [\partial_1 g](\varphi(x,y)) \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ [\partial_2 g](\varphi(x,y)) \end{pmatrix} \right\| \, |\det J_\varphi(x,y)| \, dx\, dy$$

fundamental thm for a change of variable in volume integrals

$$= \iint_\Omega \left\| \begin{pmatrix} 1 \\ 0 \\ \partial_1 g(s,t) \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ \partial_2 g(s,t) \end{pmatrix} \right\| \, ds\, dt$$

$$= \iint_\Omega \| [\partial_1 g](s,t) \wedge [\partial_2 g](s,t) \| \, ds\, dt$$

$$= \underline{\text{Area } g(\Omega)}.$$

↑ by def