Calculus II; An example of Lemma 3.19.
- Differentiability and the directional derivative -

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Lemma 3.19 (Lecture note/ [2, IV, §3, p.99]). Let $\Omega \subset \mathbb{R}^n$ be open, and let $f : \Omega \to \mathbb{R}$ be differentiable at $X_0 \in \Omega$. For any $V \in \mathbb{R}^n$ with $\|V\| = 1$ the directional derivative $D_V f(X_0)$ at X_0 exists and satisfies

$$D_V f(X_0) = V \bullet \nabla f(X_0). \tag{1}$$

Reminder

Definition 3.14 (Gradient). Let $\Omega \subset \mathbb{R}^n$ be open, and suppose that $f : \Omega \to \mathbb{R}$ admits n partial derivatives at $X \in \Omega$. Then one sets

$$\nabla f(X) = \begin{pmatrix} \partial_1 f(X) \\ \partial_2 f(X) \\ \vdots \\ \partial_n f(X) \end{pmatrix}. \tag{2}$$

Definition 3.17 (Directional derivative). Let $\Omega \subset \mathbb{R}^n$ be open, and let $f : \Omega \to \mathbb{R}$, and consider $V \in \mathbb{R}^n$ with $\|V\| = 1$. For any $X \in \Omega$ one sets

$$D_V f(X) := \lim_{\epsilon \to 0} \frac{f(X + \epsilon V) - f(X)}{\epsilon}.$$

If this limit exists, it is called the derivative of f in the direction V at X, or simply directional derivative.

Example ([1, Example 3.1, p.300]) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function $f(x_1, x_2) := e^{-x_1^2 - x_2^2}$. f is differentiable on $\mathbb{R}^2$. Let $X_0 = (x_{10}, x_{20})$ be a point in $\mathbb{R}^2$. Let $V = (v_1, v_2)$ be a located vector

(Definition 1.11, p.9) starting at X_0 with $\|V\| = \sqrt{v_1^2 + v_2^2} = 1$. Then

$$
\begin{aligned}
D_V f(X_0) &= \lim_{\epsilon\to 0} \frac{f(X_0+\epsilon V) - f(X_0)}{\epsilon} && (3)\\
&= \lim_{\epsilon\to 0} \frac{e^{-(x_{10}+\epsilon v_1)^2-(x_{20}+\epsilon v_2)^2} - e^{-x_{10}^2-x_{20}^2}}{\epsilon} && (4)\\
&= \lim_{\epsilon\to 0} \frac{e^{-x_{10}^2-2\epsilon x_{10}v_1-\epsilon^2 v_1^2-x_{20}^2-2\epsilon x_{20}v_2-\epsilon^2 v_2^2} - e^{-x_{10}^2-x_{20}^2}}{\epsilon} && (5)\\
&= \lim_{\epsilon\to 0} \frac{(e^{-x_{10}^2-x_{20}^2})(e^{-2\epsilon x_{10}v_1-2\epsilon x_{20}v_2}e^{-\epsilon^2 v_1^2-\epsilon^2 v_2^2}) - 1)}{\epsilon} && (6)
\end{aligned}
$$

Since $v_1^2 + v_2^2 = 1$,

$$
= \lim_{\epsilon\to 0} \frac{e^{-2(x_{10}v_1+x_{20}v_2)\epsilon}e^{-\epsilon^2} - 1}{e^{x_{10}^2+x_{20}^2}\epsilon} \tag{7}
$$

Since $\lim_{\epsilon\to 0}(e^{-2(x_{10}v_1+x_{20}v_2)\epsilon}e^{-\epsilon^2} - 1) = 0$, and $\lim_{\epsilon\to 0}(e^{x_{10}^2+x_{20}^2}\epsilon) = 0$, from de L'Hospital's rule,

$$
\begin{aligned}
&= \lim_{\epsilon\to 0} \frac{(-2(x_{10}v_1 + x_{20}v_2)e^{-2(x_{10}v_1+x_{20}v_2)\epsilon}e^{-\epsilon^2} - e^{-2(x_{10}v_1+x_{20}v_2)\epsilon}2\epsilon e^{-\epsilon^2}}{e^{x_{10}^2+x_{20}^2}} && (8)\\
&= \frac{-2(x_{10}v_1 + x_{20}v_2)}{e^{x_{10}^2+x_{20}^2}}. && (9)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
V \bullet \nabla f(X_0) &= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \bullet \begin{pmatrix} \partial_1 f(x_{10}, x_{20}) \\ \partial_2 f(x_{10}, x_{20}) \end{pmatrix} && (10)\\
&= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \bullet \begin{pmatrix} -2x_{10}e^{-(x_{10}^2+x_{20}^2)} \\ -2x_{20}e^{-(x_{10}^2+x_{20}^2)} \end{pmatrix} && (11)\\
&= -2v_1x_{10}e^{-(x_{10}^2+x_{20}^2)} - 2v_2x_{20}e^{-(x_{10}^2+x_{20}^2)} && (12)\\
&= \frac{-2(x_{10}v_1 + x_{20}v_2)}{e^{x_{10}^2+x_{20}^2}}. && (13)
\end{aligned}
$$

Therefore

$$
D_V f(X_0) = V \bullet \nabla f(X_0).
$$

□

For example, let $X_0 = (\cos(\frac{\pi}{4}), \sin(\frac{\pi}{4}))$ be a point in $\mathbb{R}^2$ and let $V = (\cos(\frac{\pi}{6}), \sin(\frac{\pi}{6}))$ be a located vector starting at X_0 with $\|V\| = 1$. Since f is differentiable, next calculations hold

$$
\begin{aligned}
D_V f(X_0) &= \lim_{\epsilon\to 0} \frac{e^{-(\cos(\frac{\pi}{4})+\epsilon\cos(\frac{\pi}{6}))^2-(\sin(\frac{\pi}{4})+\epsilon\sin(\frac{\pi}{6}))^2} - e^{-\cos^2(\frac{\pi}{4})-\sin^2(\frac{\pi}{4})}}{\epsilon} && (14)\\
&= -2e^{-\cos^2(\frac{\pi}{4})-\sin^2(\frac{\pi}{4})}\left(\cos(\frac{\pi}{4})\cos(\frac{\pi}{6}) + \sin(\frac{\pi}{4})\sin(\frac{\pi}{6})\right). && (15)\\
&= -2e^{-1}\cos(\frac{\pi}{12}) && (16)
\end{aligned}
$$

$$V \bullet \nabla f(X_0) = \begin{pmatrix} \cos(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) \end{pmatrix} \bullet \begin{pmatrix} -2\cos(\frac{\pi}{4})e^{-\cos^2(\frac{\pi}{4})-\sin^2(\frac{\pi}{4})} \\ -2\sin(\frac{\pi}{4})e^{-\cos^2(\frac{\pi}{4})-\sin^2(\frac{\pi}{4})} \end{pmatrix} \tag{17}$$

$$= -2e^{-1}\left(\cos(\frac{\pi}{4})\cos(\frac{\pi}{6}) + \sin(\frac{\pi}{4})\sin(\frac{\pi}{6})\right). \tag{18}$$

$$= -2e^{-1}\cos(\frac{\pi}{12}) \tag{19}$$

An illustration of this example

Out[23]=

$$e^{-x^2-y^2},\ X_0=(x_{10},y_{20})=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$$

$$V=(v_1,v_2)=(\frac{\sqrt{3}}{2},\frac{1}{2})$$

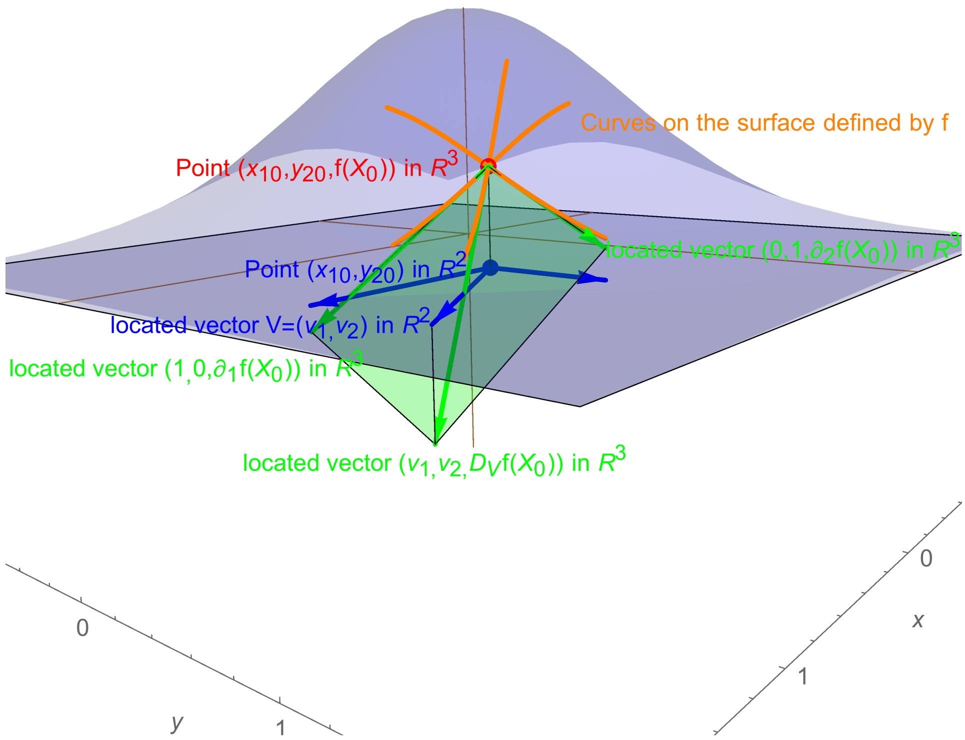

FIG. 1 $f(x_1,x_2) = e^{-x_1^2-x_2^2}$, $X_0 = (x_{10}, x_{20}) = (\cos(\frac{\pi}{4}), \sin(\frac{\pi}{4}))$, $V = (v_1, v_2) = (\cos(\frac{\pi}{6}), \sin(\frac{\pi}{6}))$.

In $\mathbb{R}^2$ fix a point X_0 and a located vector $V = (v_1, v_2)$ starting at X_0. (Blue coloured)
In $\mathbb{R}^3$ there are a point $(x_{10}, x_{20}, f(X_0))$ and three curves passing through the point in the direction of x_1 axis, x_2 axis and $V = (v_1, v_2)$. (Orange coloured)

Located vectors $(1, 0, \partial_1 f(X_0))$, $(0, 1, \partial_2 f(X_0))$ and $(v_1, v_2, D_V f(X_0))$ starting at $(x_{10}, x_{20}, f(X_0))$ are tangent to the curves respectively. (Green coloured)
The vector $(v_1, v_2, D_V f(X_0))$ is a linear combination of the vectors $(1, 0, \partial_1 f(X_0))$ and $(0, 1, \partial_2 f(X_0))$, and lies on the tangent plane at this point spanned by the two vectors.

$$(v_1, v_2, D_V f(X_0)) = v_1(1, 0, \partial_1 f(X_0)) + v_2(0, 1, \partial_2 f(X_0)).$$

Thus, FIG.1 illustrates the equation (1) in lemma 3.19.

References

[1] E.Hairer and G.Wanner, Analysis by its History, Springer 2008.
[2] Serge Lang, Calculus of Several Variables Third Edition, Springer 1987.