

### Homework 7

**Exercise 1 (Exercise 4 of HW6)** Consider the vector field  $F : \mathbb{R}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}^2$  defined for  $(x, y) \neq (0,0)$  by

$$F(x, y) = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

(i) Represent graphically this vector field (you can use polar coordinates),

(ii) Can you find a potential function for this vector field, and if so exhibit it.

**Exercise 2** a) Let  $\Omega \subset \mathbb{R}^2$  be open and let  $F : \Omega \rightarrow \mathbb{R}^2$  be of class  $C^1$  on  $\Omega$ . Let us also set  $F = (f_1, f_2)$ . Show that if

$$\partial_x f_2 \neq \partial_y f_1$$

then  $F$  does not admit a potential function of class  $C^2$ .

b) What would be a similar statement for a function  $F : \Omega \rightarrow \mathbb{R}^3$  if  $\Omega$  is an open subset of  $\mathbb{R}^3$ .

c) What about the  $n$ -dimensional case, and how many conditions have to be satisfied ?

**Exercise 3 (Spherical coordinates)** Consider the map  $\Phi : [0, \infty) \times [0, 2\pi) \times [0, \pi) \rightarrow \mathbb{R}^3$  with

$$\Phi(r, \theta, \varphi) := (r \cos(\theta) \sin(\varphi), r \sin(\theta) \sin(\varphi), r \cos(\varphi)).$$

Compute the Jacobian matrix corresponding to this function.

**Exercise 4** Let us consider  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  defined by  $f(x, y, z) = e^{xy} \cos(z)$  for any  $(x, y, z) \in \mathbb{R}^3$ . Assume also that  $x = tu$ ,  $y = \sin(tu)$  and  $z = u^2$  for some  $t, u \in \mathbb{R}$ . By setting

$$F(t, u) := f(tu, \sin(tu), u^2)$$

Compute the derivative  $\partial_2 F \equiv \partial_u F$  by three different methods: once by a direct computation, once as one component of the derivative of the composition of two functions (chain rule), and once with the formula often seen in the literature

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u}.$$

Can you explain where this formula comes from ?