

Homework 6

Exercise 1 Let us consider $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ sufficiently differentiable on \mathbb{R}^3 . One also sets $F = {}^T(F_1, F_2, F_3)$ and

$$\text{grad}(f) \equiv \nabla f = {}^T(\partial_1 f, \partial_2 f, \partial_3 f)$$

$$\text{div}(F) \equiv \nabla \cdot F = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$$

$$\text{curl}(F) \equiv \nabla \times F = {}^T(\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1)$$

$$\Delta f = \partial_1^2 f + \partial_2^2 f + \partial_3^2 f.$$

Show then the following relations:

(i) $\text{div}(fF) = f \text{div}(F) + \text{grad}(f) \cdot F,$

(ii) $\text{div}(\text{curl}(F)) = 0,$

(iii) $\text{curl}(\text{grad}(f)) = 0,$

(iv) $\text{curl}(\text{curl}(F)) = \text{grad}(\text{div}(F)) - \Delta F,$ with $\Delta F = {}^T(\Delta F_1, \Delta F_2, \Delta F_3).$

In the following exercises we call a *vector field* a function f defined on an open set $\Omega \subset \mathbb{R}^n$ and taking values in \mathbb{R}^n . In other words, for $\Omega \subset \mathbb{R}^n$ open, a function $f : \Omega \rightarrow \mathbb{R}^d$ is a vector field if $d = n$.

Exercise 2 Provide a picture for the following vector fields:

(i) $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad f(x, y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$

(ii) $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad g(x, y) = x E_1 + y E_2,$

(iii) $\nabla k : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ for $k : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $k(x, y) = \frac{1}{1+x^2+y^2}.$

Let $f : \Omega \rightarrow \mathbb{R}^n$ be a vector field. If there exists a differentiable function $\phi : \Omega \rightarrow \mathbb{R}$ such that $f = \nabla \phi$ we say that ϕ is a *potential function* for f .

Exercise 3 For the following functions, does a potential function exist ?

(i) $f(x, y) = (y, x),$ (ii) $f(x, y) = (3x^2y + 2x + y^3, x^3 + 3xy^2 - 2y),$

(iii) $f(x, y) = (\cos(x), \sin(y)),$ (iv) $f(x, y, z) = (x^2 - yz, y^2 - zx, z^2 - xy).$

Exercise 4 Consider the vector field $F : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}^2$ defined for $(x, y) \neq (0, 0)$ by

$$F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

(i) Represent graphically this vector field (you can use polar coordinates),

(ii) Can you find a potential function for this vector field, and if so exhibit it.