Nagoya University, G30 program Spring 2020

Mathematics Tutorial IIa ⚽ Instructor : Serge Richard

Homework 4

Exercise 1 (Second part of Exercise 4 of HW 3) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^1 and let $X \in \mathbb{R}^n$ such that $[\nabla f](X) \neq \mathbf{0}$. Let $k \in \mathbb{R}$ be given by $k := f(X)$ and consider the k-level set L_k. This k-level set can be considered (at least locally) as a surface of dimension $n-1$ in $\mathbb{R}^n$. Show that $[\nabla f](X)$ is perpendicular to the surface L_k. For that purpose, we can consider any parametric curve $\varphi : (-1,1) \to L_k$ with $\varphi(0) = X$ and show that $[\nabla f](X)$ is perpendicular to it at the point X.*

Exercise 2 *(i) Compute the Taylor expansion around $(0,0)$ and up to the second order of the function*

$$\mathbb{R}^2 \ni (x,y) \mapsto \mathrm{e}^{x^2+xy+y^2} \in \mathbb{R}.$$

(ii) Compute the Taylor expansion around $(0,0)$ up to the third order of the function

$$\mathbb{R}^2 \ni (x,y) \mapsto \mathrm{e}^{x+y} \in \mathbb{R}.$$

By fixing then $x = y = 1/2$ in the polynomial you have obtained, what can you say about the number e *?*

Exercise 3 *Consider the functions $\varphi : \mathbb{R} \to \mathbb{R}^2$ defined by $\varphi(t) := \binom{\cos(t)}{t^2}$, and $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) := \mathrm{e}^{3x+2y}$. We consider the composition of these two functions, namely $F : \mathbb{R} \to \mathbb{R}$ given by $F = f \circ \varphi$. Compute the derivative of this function by two different methods: once by a direct computation, and once as the derivative of a composed function (chain rule).*

So far we have considered the norm in $\mathbb{R}^n$ provided by the formula

$$\|X\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{j=1}^n x_j^2} =: \|X\|_2.$$

We could have considered other expressions, as for example

$$\|X\|_1 := |x_1| + |x_2| + \cdots + |x_n| = \sum_{j=1}^n |x_j|, \quad \text{or} \quad \|X\|_\infty := \max_{j \in \{1,\dots,n} |x_j|.$$

Exercise 4 *a) Show that the two alternative expressions also define norms. Namely, they satisfy the three conditions of any norm $|||\cdot|||$:*

(i) $|||X||| \geq 0$ for any $X \in \mathbb{R}^n$, and $|||X||| = 0$ if and only if $X = \mathbf{0}$,

(ii) $|||\lambda X||| = |\lambda|\,|||X|||$, for any $\lambda \in \mathbb{R}$ and $X \in \mathbb{R}^n$,

(iii) $|||X+Y||| \leq |||X||| + |||Y|||$, for any $X, Y \in \mathbb{R}^n$.

b) Draw the unit ball of $\mathbb{R}^2$ with the three norms $\|\cdot\|_2$, $\|\cdot\|_1$ and $\|\cdot\|_\infty$.