Homework 3

Exercise 1 *1) Compute the partial derivatives of the following functions on their respective domain:*

$$a)\ f_1(x,y)=xy,\quad b)\ f_2(x,y)=(x+1)(y+3)\quad c)\ f_3(x,y)=\frac{xy}{x^2+y^2}\quad d)\ f_4(x,y)=\frac{x+y}{x-y}.$$

2) Compute also the partial derivatives of the following functions:

(i) $f:\mathbb{R}_+\times\mathbb{R}\ni(x,y)\mapsto f(x,y)=x^y\in\mathbb{R}$,

(ii) $g:(\mathbb{R}_+)^3\to\mathbb{R}$ *given by* $g(x,y,z)=x^3y^2+\sin(xz)-\ln(xyz)$.

Let us recall the polar coordinates $(r,\theta)\in[0,\infty)\times[0,2\pi)$ which allow one to give an alternative description of all the points of $\mathbb{R}^2$. The relations between the usual coordinates $(x,y)\in\mathbb{R}\times\mathbb{R}$ are

$$x=r\cos(\theta)\qquad\text{and}\qquad y=r\sin(\theta).$$

Exercise 2 *Consider the function* $f:\mathbb{R}^2\to\mathbb{R}$ *defined by*

$$f(x,y):=\begin{cases}\frac{x^3}{x^2+y^2} & \text{if}\quad (x,y)\neq(0,0)\\ 0 & \text{if}\quad (x,y)=(0,0)\end{cases}$$

1. *By using the polar coordinates, study the continuity of* f *at* $(0,0)$,
2. *Show that* f *is differentiable with continuous partial derivatives on* $\mathbb{R}^2\setminus\{(0,0)\}$,
3. *Show that* f *admits in* $(0,0)$ *derivatives in all directions,*
4. *Show that* f *is not differentiable at* $(0,0)$.

Exercise 3 *Consider the function* $f:\mathbb{R}^2\to\mathbb{R}$ *defined by*

$$f(x,y):=\begin{cases}\frac{xy(x^2-y^2)}{x^2+y^2} & \text{if}\quad (x,y)\neq(0,0)\\ 0 & \text{if}\quad (x,y)=(0,0)\end{cases}$$

and compute its second derivatives $\partial_j\partial_k f$ *for any* $j,k\in\{x,y\}$. *Are these functions continuous ?*

Exercise 4 (Geometrical interpretation of the gradient) *a) Consider the function* $f:\mathbb{R}^2\to\mathbb{R}$ *given by* $f(x,y)=x^2+4y^2$.

(i) Compute the gradient of f *at any point* (x,y),

(ii) For $k>0$, *describe the* k*-level set* L_k, *and for this level set express* y *as a function of* x *and* k,

(iii) For any $(x,y)\in\mathbb{R}^2$ *such that* $f(x,y)=k$, *show that the gradient of* f *at* (x,y) *is orthogonal to the curve described by* L_k.

b) More generally, let $f:\mathbb{R}^n\to\mathbb{R}$ *be a function of class* C^1 *and let* $X\in\mathbb{R}^n$ *such that* $[\nabla f](X)\neq\mathbf{0}$. *Let* $k\in\mathbb{R}$ *be given by* $k:=f(X)$ *and consider the* k*-level set* L_k. *This* k*-level set can be considered (at least locally) as a surface of dimension* $n-1$ *in* $\mathbb{R}^n$. *Show that* $[\nabla f](X)$ *is perpendicular to the surface* L_k. *For that purpose, we can consider any parametric curve* $\varphi:(-1,1)\to L_k$ *with* $\varphi(0)=X$ *and show that* $[\nabla f](X)$ *is perpendicular to it at the point* X.