Homework 2

Exercise 1 *Consider a parametric curve $f:\mathbb{R}\to\mathbb{R}^d$ of class C^2, and us call* the osculating plane at *t the plane passing by $f(t)$ and defined by the two vectors $f'(t)$ and $f''(t)$. Obviously this plane is well defined only if these two vectors are not parallel.*

(i) Determine the osculating plane at any t for the curve in $\mathbb{R}^3$ defined by the function

$$f:\mathbb{R}\ni t\mapsto \big(\cos(t),\sin(t),t\big)\in\mathbb{R}^3,$$

(ii) More generally, for any parametric curve f of class C^2 and for any diffeomorphism φ of class C^2, show that the osculating plane defined by f or by the new parametric curve $f\circ\varphi$ **at the same point** *are equal.*

Exercise 2 *Let $\Omega\subset\mathbb{R}^2$ and consider the functions $f_i:\Omega\to\mathbb{R}$ defined for $(x,y)\in\Omega$ by*

$$a)\ f_1(x,y)=xy,\quad b)\ f_2(x,y)=(x+1)(y+3)\quad c)\ f_3(x,y)=\frac{xy}{x^2+y^2}\quad d)\ f_4(x,y)=\frac{x+y}{x-y}.$$

1. *Determine the maximal domain Ω on which these functions are well defined,*
2. *Sketch the k-level sets for these functions.*

Exercise 3 *Consider the following functions defined on $\mathbb{R}^2\setminus\{(0,0)\}$*

$$a)\ f_1(x,y)=\frac{x^2-y^2}{x^2+y^2},\quad b)\ f_2(x,y)=\frac{xy}{x^2+y^2},\quad c)\ f_3(x,y)=\frac{1}{x^2+y^2+1}.$$

For each of them compute the limits $\lim_{x\to0}\big(\lim_{y\to0}f_i(x,y)\big)$, $\lim_{y\to0}\big(\lim_{x\to0}f_i(x,y)\big)$, and $\lim_{(x,y)\to(0,0)}f_i(x,y)$. Discuss your result.

Exercise 4 *Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by*

$$f(x,y):=\begin{cases}\frac{x^2y}{x^4-2x^2y+3y^2} & \text{if}\quad (x,y)\neq(0,0)\\ 0 & \text{if}\quad (x,y)=(0,0)\end{cases}$$

1. *Study the limit $(x,y)\to(0,0)$ along the path of equation $y=mx$ for any $m\in\mathbb{R}$,*
2. *Study the limit $(x,y)\to(0,0)$ along the path of equation $y=x^2$,*
3. *Show that f is not continuous at $(0,0)$.*