Homework 11

Exercise 1 *Let X, Y be two vectors in $\mathbb{R}^2$. Check that the area of the parallelogram spanned by X and Y is equal to the absolute value of the determinant of the matrix $(X\ Y) \in M_2(\mathbb{R})$. More generally, if $X_1, \dots, X_n$ are n vectors of $\mathbb{R}^n$, one writes $\mathsf{Vol}(X_1, \dots, X_n)$ for the volume of the n-dimensional box spanned by $X_1, \dots, X_n$. Why is it natural to have*

$$\mathsf{Vol}(X_1, \dots, X_n) = |\,\mathsf{Det}(X_1 \dots X_n)|\ ?$$

Exercise 2 *Use Green's theorem to compute the integral $\int_c f$ with $f(x, y) = (y^2, x)$ when c corresponds to the following curves, taken counterclockwise:*

(i) The square of vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$,

(ii) The square of vertices $(\pm 1, \pm 1)$,

(iii) The circle of radius 1 and centered at $(0, 0)$,

(iv) The ellipse of equation $(x/a)^2 + (y/b)^2 = 1$ for some $a, b > 0$.

Exercise 3 *Check the validity of Green's theorem for the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined for $(x, y) \in \mathbb{R}^2$ by $f(x, y) = (2xy, x^2)$ on the domain $\Omega = [-1, 2] \times [-1, 3] \subset \mathbb{R}^2$.*

Exercise 4 *Compute the area of the domain defined in $\mathbb{R}_+ \times \mathbb{R}_+$ by the four curves of equation*

$$y = ax, \quad y = x/a, \quad y = b/x, \quad y = 1/bx \qquad \text{for } a > 1, b > 1.$$

Let Ω be a body in $\mathbb{R}^n$ and let $\varrho : \Omega \to \mathbb{R}_+$ denote its density function (for $X \in \Omega$ the value $\varrho(X)$ denotes the density of Ω at X). Let M denote the total mass of the body, and let $\bar{X}$ denote the coordinates of its center of mass. These quantities are defined by

$$M = \int_\Omega \varrho(X) \mathrm{d}X,$$
$$\bar{X} = \frac{1}{M} \int_\Omega X \varrho(X) \mathrm{d}X.$$

Note that the second line represents in fact n equalities.

Exercise 5 (For a rainy day) *(i) Find the center of mass of the quarter of a unit disc Ω defined in polar coordinates by*

$$\Omega = \big\{\big(r\cos(\theta), r\sin(\theta)\big) \in \mathbb{R}^2 \mid 0 \le r \le 1 \text{ and } 0 \le \theta \le \pi/2\big\},$$

(ii) Find the z-coordinate of the center of mass of the upper half of a unit ball centered at $0 \in \mathbb{R}^3$.