An exercise of Cal II lecture 1

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April 11, 2020

Lemma 1.17. For any $X, Y \in \mathbb{R}^n$,

(i) $|X \cdot Y| \leq \|X\|\|Y\|$,

(ii) $\|X + Y\| \leq \|X\| + \|Y\|$,

Proof (I wrote (i) by reference to [1], Chapter 1, p.2-3.)
Let $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$.

(i) For $n = 1$, the inequality is trivially true. For $n = 2$, the inequality just says

$$(x_1y_1 + x_2y_2)^2 \leq (x_1^2 + x_2^2)(y_1^2 + y_2^2). \tag{1}$$

The inequality (1) is equivalent to

$$x_1^2y_1^2 + 2x_1y_1x_2y_2 + x_2^2y_2^2 \leq x_1^2y_1^2 + x_1^2y_2^2 + x_2^2y_1^2 + x_2^2y_2^2, \tag{2}$$

$$0 \leq (x_1y_2)^2 - 2(x_1y_2)(x_2y_1) + (x_2y_1)^2. \tag{3}$$

Since the right hand side of (3) is

$$(x_1y_2)^2 - 2(x_1y_2)(x_2y_1) + (x_2y_1)^2 = (x_1y_2 - x_2y_1)^2. \tag{4}$$

The nonnegativity of (4) confirms the truth of inequality (2). By our chain of equivalences, we find that the inequality (1) is also true, and thus we have proved the inequality (i) for $n = 2$.
Let's $H(n)$ stand for the hypothesis that the inequality is valid for n. Applying $H(n)$ and then using $H(2)$, we have

$$\begin{aligned}
&x_1y_1 + x_2y_2 + \cdots + x_ny_n + x_{n+1}y_{n+1} \\
&\qquad = (x_1y_1 + x_2y_2 + \cdots + x_ny_n) + x_{n+1}y_{n+1} \\
&\qquad \leq (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}(y_1^2 + y_2^2 + \cdots + y_n^2)^{\frac{1}{2}} + x_{n+1}y_{n+1} \\
&\qquad \leq (x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^2)^{\frac{1}{2}}(y_1^2 + y_2^2 + \cdots + y_n^2 + y_{n+1}^2)^{\frac{1}{2}},
\end{aligned}$$

where in the first inequality we used $H(n)$, and in the second inequality we used $H(2)$ in the form

$$\alpha\beta + x_{n+1}y_{n+1} \leq (\alpha^2 + x_{n+1}^2)^{\frac{1}{2}}(\beta^2 + y_{n+1}^2)^{\frac{1}{2}}$$

with the new variables

$$\alpha = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}} \quad \text{and} \quad \beta = (y_1^2 + y_2^2 + \cdots + y_n^2)^{\frac{1}{2}}.$$

□

(ii) The inequality (ii) just says

$$(X + Y) \cdot (X + Y) \leq X \cdot X + Y \cdot Y + 2\|X\|\|Y\|. \tag{5}$$

The inequality (5) is equivalent to

$$(x_1 + y_1)^2 + (x_2 + y_2)^2 + \cdots + (x_n + y_n)^2 \leq x_1^2 + x_2^2 + \cdots + x_n^2 + y_1^2 + y_2^2 + \cdots + y_n^2 + 2\|X\|\|Y\|, \tag{6}$$

$$2x_1y_1 + 2x_2y_2 + \cdots + 2x_ny_n \leq 2\|X\|\|Y\|, \tag{7}$$

$$X \cdot Y \leq \|X\|\|Y\|. \tag{8}$$

Since $X \cdot Y \leq |X \cdot Y|$, from the inequality (i), the inequality (8) is true, and this confirms the truth of inequality (6). By our chain of equivalences, we find that the inequality (5) is also true, and thus we have proved the inequality (ii).

□

Reference

[1] J. Michael Steele, The Cauchy-Schwarz Master Class, Cambridge University press, 2004.