Chapter 9 Zhengliang Zhu

9.16.

a. $H_0: \theta = \theta_0$ $H_1: \theta \neq \theta_0$

α: we have the rejection region

$|\bar{x} - \theta_0| \geq Z_{\alpha/2}\, \sigma/\sqrt{n}$

$\Leftrightarrow$ acceptance region is $\{x \mid \theta_0 - Z_{\alpha/2}\sigma/\sqrt{n} \leq \bar{x} \leq \theta_0 + Z_{\alpha/2}\sigma/\sqrt{n}\}$

$1-\alpha$: the confidence interval is $|\bar{x} - \theta| \leq Z_{\alpha/2}\sigma/\sqrt{n}$

$\{\theta \mid \bar{x} - Z_{\alpha/2}\sigma/\sqrt{n} \leq \theta \leq \bar{x} + Z_{\alpha/2}\sigma/\sqrt{n}\}$

b. $H_0: \theta \geq \theta_0$, $H_1: \theta < \theta_0$.

the rejection region is $\bar{x} - \theta_0 < -Z_\alpha \sigma/\sqrt{n}$

α. the acceptance region is $\{x \mid \bar{x} - \theta_0 \geq -Z_\alpha \sigma/\sqrt{n}\}$

$1-\alpha$: the confidence interval $\{\theta \mid \theta \leq \bar{x} + Z_\alpha \sigma/\sqrt{n}\}$

c. the rejection region is $\bar{x} - \theta_0 > Z_\alpha \sigma/\sqrt{n}$

the acceptance region is $\{x \mid \bar{x} - \theta_0 \leq Z_\alpha \sigma/\sqrt{n}\}$

the confidence interval is $\{\theta : \theta \geq \bar{x} - Z_\alpha \sigma/\sqrt{n}\}$

9.17

a) $\theta - \frac{1}{2} < X < \theta + \frac{1}{2}$

$-\frac{1}{2} < X - \theta < \frac{1}{2}$

let $P(x_1 \le X - \theta \le x_2) = x_2 - x_1 = 1 - \alpha$.

then one choice is

$x_2 = \frac{1}{2} - \frac{1}{2}\alpha$

$x_1 = -\frac{1}{2} + \frac{1}{2}\alpha$.

b)

$f(x|\theta) = 2x/\theta^2$

$\int f\,dx = \int 2\frac{x}{\theta}\,d\frac{x}{\theta} = \int 2t\,dt \quad (t = \frac{x}{\theta})$

$0 < x < \theta,\ \theta > 0 \Rightarrow 0 < \frac{x}{\theta} < 1$

we let $P(x_1 \le \frac{x}{\theta} \le x_2) = \int_{x_1}^{x_2} 2t\,dt = x_2^2 - x_1^2 = 1 - \alpha$.

because $0 < \frac{x}{\theta} < 1$

so $x_2^2 = 1 - \frac{\alpha}{2}$, $x_1^2 = \frac{\alpha}{2}$

$x_2 = \sqrt{1 - \frac{\alpha}{2}}$, $x_1 = \sqrt{\frac{\alpha}{2}}$

11.35

a) let $\frac{d}{d\theta} \sum_{i=1}^{n} (Y_i - \theta x_i^2)^2 = 0$

$= 2\sum_{i=1}^{n} (Y_i - \theta x_i^2)(-x_i^2) = 0$

$\Leftrightarrow \sum_{i=1}^{n} \theta x_i^4 = \sum_{i=1}^{n} x_i^2 Y_i$

$\Leftrightarrow \hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2 Y_i}{\sum_{i=1}^{n} x_i^4}$

b) $\epsilon \sim n(0, \sigma^2)$ $\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$ $\epsilon = Y_i - \theta x_i^2$

$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(Y_i - \theta x_i^2)^2}{2\sigma^2}}$

$\frac{\partial \ln L}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\ln\left((2\pi\sigma^2)^{-\frac{n}{2}}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \theta x_i^2)^2\right)$

$= \frac{\partial}{\partial\theta}\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n} 2(Y_i - \theta x_i^2)(-x_i)^2 = 0\right.$

$\Leftrightarrow \hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2 Y_i}{\sum_{i=1}^{n} x_i^4}$

C.

$$E_\theta(\hat{\theta}) = E_\theta\left(\frac{\sum_{i=1}^{n} y_i x_i^2}{\sum_{i=1}^{n} x_i^4}\right) \qquad E(y_i - \theta x_i^2) = 0$$

x_i is constant

$E_\theta(\hat{\theta}) - \theta = \text{Bias}$

$E(y_i) = x_i^2 E(\theta)$

$$= \frac{1}{\sum_{i=1}^{n} x_i^4} \sum_{i=1}^{n} x_i^2 E_\theta(y_i) - \theta = \frac{\sum_{i=1}^{n} x_i^4 \theta}{\sum_{i=1}^{n} x_i^4} - \theta = \theta - \theta = 0$$

so $E_\theta(\hat{\theta}) = \theta$. $\hat{\theta}$ is unbiased estimator.

And:

The derivatives of the log likelihood are

$$\frac{d}{d\theta}\log L = \frac{1}{\sigma^2}\sum_i (y_i - \theta x_i^2) x_i^2$$

$$\frac{d^2}{d\theta^2}\log L = \frac{-1}{\sigma^2}\sum_i x_i^4.$$

so the CRLB is $\sigma^2/\sum_i x_i^4$. The variance of $\hat{\theta}$ is

$$\operatorname{Var}\hat{\theta} = \operatorname{var}\left(\frac{\sum_i y_i x_i^2}{\sum_i x_i^4}\right) = \sum_i \left(\frac{x_i^2}{\sum_j x_j^4}\right)\sigma^2 = \sigma^2/\sum_i x_i^4$$

so $\hat{\theta}$ is the best unbiased estimator.