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Exercise 1.3.2 (Re)

Let $\mathcal{M}_1, \mathcal{M}_2$ be two dense lin. mfd. of $\mathcal{H}$, and let $B \in \mathcal{B}(\mathcal{H})$. Then
$\|B\| = \sup_{\substack{f \in \mathcal{M}_1, g \in \mathcal{M}_2 \\ \|f\|=1, \|g\|=1}} |\langle f, Bg \rangle|$.

$\because$)

Note that, by definition, $\|B\| = \sup_{f \neq 0} \frac{\|Bf\|}{\|f\|} = \sup_{\|f\|=1} \|Bf\|$, and $\|Bf\| \leq \|B\| \cdot \|f\|$ for all $f \in \mathcal{H}$. Then, (i) $\{f_n\}_n \subset \mathcal{H}$: Cauchy $\Rightarrow$ $\{Bf_n\}_n \subset \mathcal{H}$: Cauchy ; (ii) $s\text{-}\lim f_n = f_\infty \Rightarrow s\text{-}\lim Bf_n = Bf_\infty$.

Let $g \in \mathcal{H}$ with $\|g\| = 1$. Since $\mathcal{M}_2 \subset \mathcal{H}$ is dense, there exists a sequence $\{g_n\}_n \subset \mathcal{M}_2$ s.t. $s\text{-}\lim g_n = g$. We can assume that $\|g_n\| = 1$: Indeed,

$$
\begin{aligned}
\left\| \frac{g_n}{\|g_n\|} - g \right\| &= \| (g_n - g) + (1 - \|g_n\|) g \| / \|g_n\| \\
&\leq (\|g_n - g\| + |1 - \|g_n\|| \cdot \|g\|) / \|g_n\| \\
&\leq (\|g_n - g\| + \|g - g_n\|) / \|g_n\| \quad (\because \|g\|=1 \text{ and } (1.5)). \\
&= 2\|g_n - g\| / \|g_n\| \longrightarrow 2 \cdot 0 / \|g\| = 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Then above observation yields $s\text{-}\lim Bg_n = Bg$, and hence $\lim \|Bg_n\| = \|Bg\|$. Therefore, we obtain $\|B\| = \sup_{g \in \mathcal{M}_2, \|g\|=1} \|Bg\|$.

Now, let $h \in \mathcal{H}$, $h \neq 0$. Then if $\|f\| = 1$, $|\langle f, h \rangle| \leq \|f\| \cdot \|h\| = \|h\|$ with equality for $f = h / \|h\|$. Thus $\|h\| = \sup_{\|f\|=1} |\langle f, h \rangle|$, and by the same argument above, we have $\|h\| = \sup_{f \in \mathcal{M}_1, \|f\|=1} |\langle f, h \rangle|$ for all $h \in \mathcal{H}$.

Hence, we obtain

$$
\|B\| = \sup_{g \in \mathcal{M}_2, \|g\|=1} \|Bg\| = \sup_{g \in \mathcal{M}_2, \|g\|=1} \sup_{f \in \mathcal{M}_1, \|f\|=1} |\langle f, Bg \rangle|. \quad \boxed{///}
$$