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Theorem 5.4.4 about order statistics:

Let $X_{(1)}, \dots, X_{(n)}$ denote the order statistics of a random sample $X_1; \dots; X_n$, from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$. Then the pdf of $X_{(j)}$ is:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!\,(n-j)!} f_X(x) \,[F_X(x)]^{j-1} [1 - F_X(x)]^{n-j}$$

Proof:

⊕ Let Y be a random variable that counts the number of $X_1, \dots, X_n$ that are less than or equal to x:

$$Y := \# \{X_j \leqslant x \text{ for } j = 1; \dots; n\}$$

Y is discrete.

⊕ For each of $X_1, \dots, X_n$, we define a success as the event $\{X_j \leqslant x\}$. Then Y is the number of successes in n trials.

$\left(\begin{array}{l} (X_1, \dots, X_n) \text{ is} \\ \text{a random sample} \end{array} \right) \underset{\text{By def.}}{\Rightarrow} \left\{ \begin{array}{l} X_1, \dots, X_n \text{ are identically distributed} \Rightarrow \text{probability of a success is the same value for each trial, which is } F_X(x) = P(X_j \leqslant x) \; j \in \{1; \dots; n\} \\ X_1, \dots, X_n \text{ are independent} \Rightarrow \text{success of } j^{th} \text{ trial is independent of outcomes of other trials} \end{array} \right.$

$\Rightarrow Y \sim \text{binomial } (n, F_X(x))$

⊕ Let $X_{(1)}, \dots, X_{(n)}$ be the order statistics of the random sample $X_1, \dots, X_n$.

The event $\{X_{(j)} \leqslant x\}$ is equivalent to the event $\{Y \geqslant j\}$ since they have the same meaning : at least j of the sample values are less than or equal to x.

$$\Rightarrow F_{X_{(j)}}(x) = P(X_{(j)} \leqslant x) = P(Y \geqslant j) \underset{(Y \sim \text{binomial } (n, F_X(x)))}{=} \sum_{k=j}^{n} \binom{n}{k} [F_X(x)]^k [1 - F_X(x)]^{n-k} \quad \text{with } \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

⊕ Differentiating the cdf to obtain the pdf :

$$f_{X_{(j)}}(x) = \sum_{k=j}^{n} \binom{n}{k} \Big\{ k [F_X(x)]^{k-1} f_X(x) [1 - F_X(x)]^{n-k} + [F_X(x)]^k (n-k) [1 - F_X(x)]^{n-k-1} (-f_X(x))$$

$$= \binom{n}{j} j [F_X(x)]^{j-1} f_X(x) [1 - F_X(x)]^{n-j} + \sum_{k=j+1}^{n} \binom{n}{k} k f_X(x) [F_X(x)]^{k-1} [1 - F_X(x)]^{n-k}$$

$$- \sum_{k=j}^{n} \binom{n}{k} (n-k) f_X(x) [F_X(x)]^k [1 - F_X(x)]^{n-k-1}$$

(change dummy variable)

$$= \binom{n}{j} j f_X(x) [F_X(x)]^{j-1} [1 - F_X(x)]^{n-j} + \sum_{k=j}^{n-1} \binom{n}{k+1} (k+1) f_X(x) [F_X(x)]^k [1 - F_X(x)]^{n-k-1}$$

$$- \sum_{k=j}^{n-1} \binom{n}{k} (n-k) f_X(x) [F_X(x)]^k [1 - F_X(x)]^{n-k-1}$$

($n = k$ term is 0)

Moreover, $\binom{n}{k+1} (k+1) = \frac{n!}{(k+1)!\,(n-k-1)!} (k+1) = \frac{n!}{k!\,(n-k-1)!} = \frac{n!}{k!\,(n-k)!} (n-k) = \binom{n}{k} (n-k)$

$$\Rightarrow f_{X_{(j)}}(x) = \binom{n}{j} j f_X(x) [F_X(x)]^{j-1} [1 - F_X(x)]^{n-j}$$

$$\Leftrightarrow f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x) [F_X(x)]^{j-1} [1 - F_X(x)]^{n-j} \qquad \text{(Q.E.D)}$$