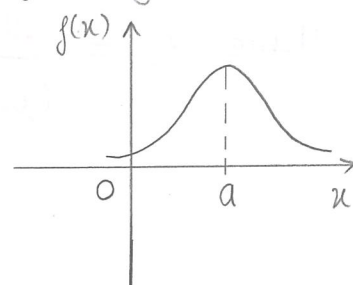


a) Firstly let us calculate the mean, given that the pdf is symmetric about a point  $a$ . One has:

$$\mu = \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^{\infty} (x+a) f(x+a) d(x+a)$$



For example, a graph of  $f(x)$  may look like this

$$\Leftrightarrow \mu = \underbrace{\int_{-\infty}^{\infty} x f(x+a) dx}_0 + a \underbrace{\int_{-\infty}^{\infty} f(x+a) d(x+a)}_1 \quad (\text{by definition of a pdf})$$

(because  $x$  is odd function about 0 and  $f(x+a)$  is even function about 0)

$$\text{So } \mu = a$$

The  $n^{\text{th}}$  central moment is then:  $\mu_n = \int_{-\infty}^{\infty} (x-a)^n f(x) dx$

$$\mu_3 = \int_{-\infty}^{\infty} (x-a)^3 f(x) dx = \int_{-\infty}^{\infty} y^3 f(y+a) dy = 0$$

(again because  $y^3$  is odd and  $f(y+a)$  is even)

$$\text{Hence } \alpha_3 := \frac{\mu_3}{(\mu_2)^{3/2}} = 0 \quad \text{for a symmetric pdf}$$

$\alpha_3$  is called the skewness, which measures the lack of symmetry of the pdf.

So it is not surprising that  $\alpha_3 = 0$  when the pdf is symmetric about a point.

b) For example let us consider  $f(x) = e^{-x}$ ,  $x \geq 0$

(Note that  $f(x)$  is not symmetric here)

Firstly, let us calculate the mean:

$$\begin{aligned} \mu &= \int_0^{\infty} x e^{-x} dx = -x e^{-x} \Big|_0^{\infty} + \int_0^{\infty} e^{-x} dx \\ &= (0-0) + (-e^{-x}) \Big|_0^{\infty} = 1 \end{aligned}$$

( $\lim_{x \rightarrow \infty} x^n e^{-x} = 0$ ,  $n \in \mathbb{N}$  using L'Hospital rule  $n$  times)

$$\begin{aligned} \text{So } \mu_2 &= \int_0^{\infty} (x-1)^2 e^{-x} dx = -(x-1)^2 e^{-x} \Big|_0^{\infty} + \int_0^{\infty} 2(x-1) e^{-x} dx \\ &= 1 - 2(x-1) e^{-x} \Big|_0^{\infty} + \int_0^{\infty} 2e^{-x} dx \\ &= 1 - 2 + 2 = 1 \end{aligned}$$

$$\begin{aligned}\mu_3 &= \int_0^{\infty} (x-1)^3 e^{-x} dx = -(x-1)^3 e^{-x} \Big|_0^{\infty} + \int_0^{\infty} 3(x-1)^2 e^{-x} dx \\ &= -1 + 3\mu_2 = -1 + 3 = 2\end{aligned}$$

$$\text{Hence } \alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} = 2$$

Exercise 2.28 (c1) Part (c) Calculate  $\alpha_4$

Liyang, Sun. Chang

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty$$

This is the pdf of standard normal distribution.

$$\Rightarrow \mu = 0, \quad \mu_2 = \sigma^2 = 1$$

$$\mu_4 = \int_{-\infty}^{\infty} (x-\mu)^4 f(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^4 e^{-\frac{x^2}{2}} dx$$

$$\sqrt{2\pi} \mu_4 = \int_{-\infty}^{\infty} x^3 x e^{-\frac{x^2}{2}} dx = -x^3 e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} 3x^2 dx$$

$$= 3 \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx = 3\sqrt{2\pi} \mu_2 = 3\sqrt{2\pi}$$

$$\Rightarrow \mu_4 = 3$$

$$\Rightarrow \alpha_4 = \frac{\mu_4}{\mu_2^2} = 3 \quad \left( \alpha_4 := \frac{\mu_4}{\mu_2^2} \text{ is called kurtosis which measures the peakedness or flatness of the pdf} \right)$$

$$f(x) = e^{-|x|} \quad (c3)$$

$$\mu_2 = \int_{-\infty}^{\infty} x^2 e^{-|x|} dx = 2 \int_0^{\infty} x^2 e^{-x} dx = 2(-x^2 e^{-x} \Big|_0^{\infty} + \int_0^{\infty} 2x e^{-x} dx) = 4(-x e^{-x} \Big|_0^{\infty} + \int_0^{\infty} e^{-x} dx) = 4.$$

$$\mu_4 = \int_{-\infty}^{\infty} x^4 e^{-|x|} dx = 2 \int_0^{\infty} x^4 e^{-x} dx = 2(-x^4 e^{-x} \Big|_0^{\infty} + 4 \int_0^{\infty} x^3 e^{-x} dx) = 8(-x^3 e^{-x} \Big|_0^{\infty} + \int_0^{\infty} 3x^2 e^{-x} dx) = 12 \cdot 2 \int_0^{\infty} x^2 e^{-x} dx = 12\mu_2 = 48$$

$$\Rightarrow \alpha_4 = \mu_4 / \mu_2^2 = 48 / 4^2 = 48 / 16 = 3.$$

