**Nagoya University, G30 program** — Spring 2019

Mathematics Tutorial IIa (calculus) — Instructor : Serge Richard

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**Homework 6**

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**Exercise 1** *Let us consider $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = \mathrm{e}^{xy}\cos(z)$ for any $(x, y, z) \in \mathbb{R}^3$. Assume also that $x = tu$, $y = \sin(tu)$ and $z = u^2$ for some $t, u \in \mathbb{R}$. By setting*

$$F(t, u) := f\big(tu, \sin(tu), u^2\big)$$

*Compute the derivative $\partial_2 F \equiv \partial_u F$ by three different methods: once by a direct computation, once as one component of the derivative of the composition of two functions (chain rule), and once with the formula often seen in the literature*

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}.$$

*Can you explain where this formula comes from ?*

**Exercise 2 (Spherical coordinates)** *Consider the map $\Phi : [0, \infty) \times [0, 2\pi) \times [0, \pi) \to \mathbb{R}^3$ with*

$$\Phi(r, \theta, \varphi) := \big(r\cos(\theta)\sin(\varphi), r\sin(\theta)\sin(\varphi), r\cos(\varphi)\big).$$

*Compute the Jacobian matrix corresponding to this function.*

Recall that *a vector field* is a function $f$ defined on an open set $\Omega \subset \mathbb{R}^n$ and taking values in $\mathbb{R}^n$. For such a vector field $f$, if there exists a differentiable function $\phi : \Omega \to \mathbb{R}$ such that $f = \nabla\phi$ we say that $\phi$ is a *potential function for $f$*.

**Exercise 3** *For the following functions, does it exist a potential function ?*

*(i)* $f(x, y) = (y, x)$,

*(ii)* $f(x, y) = (3x^2y + 2x + y^3, x^3 + 3xy^2 - 2y)$,

*(iii)* $f(x, y) = \big(\cos(x), \sin(y)\big)$,

*(iv)* $f(x, y, z) = \big(x^2 - yz, y^2 - zx, z^2 - xy\big)$.

**Exercise 4** *a) Let $\Omega \subset \mathbb{R}^2$ be open and let $F : \Omega \to \mathbb{R}^2$ be of class $C^1$ on $\Omega$. Let us also set $F = (f_1, f_2)$. Show that if*

$$\partial_x f_2 \neq \partial_y f_1$$

*then $F$ does not admit a potential function of class $C^2$.*

*b) What would be a similar statement for a function $F : \Omega \to \mathbb{R}^3$ if $\Omega$ is an open subset of $\mathbb{R}^3$.*

*c) What about the $n$-dimensional case, and how many conditions have to be satisfied ?*