Nagoya University, G30 program Spring 2019

Mathematics Tutorial IIa (calculus) Instructor : Serge Richard

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Homework 5

Exercise 1 *Draw the unit ball of* $\mathbb{R}^2$ *with the three norms* $\|\cdot\|_2$, $\|\cdot\|_1$ *and* $\|\cdot\|_\infty$.

Exercise 2 *Let us consider* $f:\mathbb{R}^3\to\mathbb{R}$ *and* $F:\mathbb{R}^3\to\mathbb{R}^3$ *sufficiently differentiable on* $\mathbb{R}^3$. *One also sets* $F={}^T(F_1,F_2,F_3)$ *and*

$$\begin{aligned}
\mathrm{grad}(f) &\equiv \nabla f = {}^T\big(\partial_1 f,\partial_2 f,\partial_3 f\big)\\
\mathrm{div}(F) &\equiv \nabla\cdot F = \partial_1F_1+\partial_2F_2+\partial_3F_3\\
\mathrm{curl}(F) &\equiv \nabla\times F = {}^T\big(\partial_2F_3-\partial_3F_2,\partial_3F_1-\partial_1F_3,\partial_1F_2-\partial_2F_1\big)\\
\Delta f &= \partial_1^2 f+\partial_2^2 f+\partial_3^2 f.
\end{aligned}$$

Show then the following relations:

(i) $\mathrm{div}(fF)=f\mathrm{div}(F)+\mathrm{grad}(f)\cdot F$,

(ii) $\mathrm{div}\big(\mathrm{curl}(F)\big)=0$,

(iii) $\mathrm{curl}\big(\mathrm{grad}(f)\big)=0$,

(iv) $\mathrm{curl}\big(\mathrm{curl}(F)\big)=\mathrm{grad}\big(\mathrm{div}(F)\big)-\Delta F$, *with* $\Delta F={}^T(\Delta F_1,\Delta F_2,\Delta F_3)$.

In the following exercises we call *a vector field* a function f defined on an open set $\Omega\subset\mathbb{R}^n$ and taking values in $\mathbb{R}^n$. In other words, for $\Omega\subset\mathbb{R}^n$ open, a function $f:\Omega\to\mathbb{R}^d$ is a vector field if $d=n$.

Exercise 3 *Provide a picture for the following vector fields:*

(i) $f:\mathbb{R}^2\to\mathbb{R}^2,\quad f(x,y)=\binom{1}{1}$,

(ii) $g:\mathbb{R}^2\to\mathbb{R}^2,\quad g(x,y)=x\,E_1+y\,E_2$,

(iii) $\nabla k:\mathbb{R}^2\to\mathbb{R}^2$ *for* $k:\mathbb{R}^2\to\mathbb{R}$ *given by* $k(x,y)=\frac{1}{1+x^2+y^2}$.

Exercise 4 *Consider the functions* $\varphi:\mathbb{R}\to\mathbb{R}^2$ *defined by* $\varphi(t):=\binom{\cos(t)}{t^2}$, *and* $f:\mathbb{R}^2\to\mathbb{R}$ *defined by* $f(x,y):=\mathrm{e}^{3x+2y}$. *We consider the composition of these two functions, namely* $F:\mathbb{R}\to\mathbb{R}$ *given by* $F=f\circ\varphi$. *Compute the derivative of this function by two different methods: once by a direct computation, and once as the derivative of a composed function (chain rule).*