Nagoya University, G30 program Spring 2019
Mathematics Tutorial IIa (calculus) Instructor : Serge Richard

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Homework 11

Exercise 1 *Compute the integral of the function* $f : \mathbb{R}^2 \to \mathbb{R}$*, defined by* $f(x, y) = x$*, on the domain* Ω *with*

$$\Omega := \big\{\big(r\cos(\theta), r\sin(\theta)\big) \in \mathbb{R}^2 \mid 0 \leq \theta \leq \pi/2 \text{ and } 0 \leq r \leq 2\cos(\theta)\big\}.$$

Exercise 2 *Find the integral of the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x, y) = \frac{1}{(x^2+y^2+1)^{3/2}}$ *on the disc of radius* R *and centered at the origin of* $\mathbb{R}^2$*.*

Exercise 3 *Compute the area enclosed by the curve given in polar coordinate by* $r^2 = \cos(\theta)$*. Sketch first this area.*

Exercise 4 *Find the mass of a spherical ball of radius* R *if the density of the ball at any point is equal to a constant* k *times the distance of that point to the center of the ball.*

Exercise 5 *Find the integral of the function* $f : \mathbb{R}^3 \to \mathbb{R}$ *given by* $f(x, y, z) = x^2$ *over the portion of the cylinder defined by* $x^2 + y^2 = a^2$ *and lying between the planes defined by* $z = 0$ *and* $z = b$*, with* $a > 0$ *and* $b > 0$*.*

Let Ω be a body in $\mathbb{R}^n$ and let $\varrho : \Omega \to \mathbb{R}_+$ denote its density function (for $X \in \Omega$ the value $\varrho(X)$ denotes the density of Ω at X). Let M denote the total mass of the body, and let $\bar{X}$ denote the coordinates of its center of mass. These quantities are defined by

$$M = \int_\Omega \varrho(X)\mathrm{d}X,$$
$$\bar{X} = \frac{1}{M}\int_\Omega X\varrho(X)\mathrm{d}X.$$

Note that the second line represents in fact n equalities.

Exercise 6 *(i) Find the center of mass of the quarter of a unit disc* Ω *defined in polar coordinates by*

$$\Omega = \big\{\big(r\cos(\theta), r\sin(\theta)\big) \in \mathbb{R}^2 \mid 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq \pi/2\big\},$$

(ii) Find the z*-coordinate of the center of mass of the upper half of a unit ball centered at* $0 \in \mathbb{R}^3$*.*

Exercise 7 *Let* X, Y *be two vectors in* $\mathbb{R}^2$*. Check that the area of the parallelogram spanned by* X *and* Y *is equal to the absolute value of the determinant of the matrix* $(X\ Y) \in M_2(\mathbb{R})$*. More generally, if* $X_1, \dots, X_n$ *are* n *vectors of* $\mathbb{R}^n$*, one writes* $\mathsf{Vol}(X_1, \dots, X_n)$ *for the volume of the* n*-dimensional box spanned by* $X_1, \dots, X_n$*. Why is it natural to have*

$$\mathsf{Vol}(X_1, \dots, X_n) = |\,\mathsf{Det}(X_1 \dots X_n)|\ ?$$