By inspection, the possible pairs (G_+, G) are

$$\left.\begin{array}{ll} (C_n, C_{2n}) & \text{for } n \in \{1,2,3\} \\ (C_n, D_{2n}) & \quad '' \\ (D_n, D_{2n}) & \quad '' \\ (T, O) & \end{array}\right\} 10 \text{ solutions}$$

Implicit: Column 1 are subgroups of column 2

Now we have all 32 finite subgroups of $O(3)$ which leave a lattice invariant

Next step: for a given subgroup, find the corresponding invariant lattice

$\rightarrow$ 7 lattice systems

14 Bravais lattices

Exercise: Do the same thing with $E(3)$ (containing translation) instead of $O(3)$

$\rightarrow$ 230 finite subgroups leaving a lattice invariant.

Or for $O(2)$

Chapter II: Linear representation

II.1 Generalities

Def. A Hilbert space is a complex vector space together with an inner product $\langle \cdot , \cdot \rangle$ (linear in the second argument) and complete for the norm $\|f\| = \sqrt{\langle f, f \rangle}$

Example: ($\mathbb{R}^n$ "real Hilbert space")

- $\mathbb{C}^n$ with $a = (a_1, \cdots, a_n), b = (b_1, \cdots, b_n) \in \mathbb{C}^n$ $\langle a, b \rangle = \sum_{j=1}^{n} \overline{a_j} \, b_j$
- $L^2(\mathbb{R}^n)$ with $\langle f, g \rangle = \int_{\mathbb{R}^n} \overline{f(x)} \, g(x) \, dx$
- $\ell^2(\mathbb{Z}^n)$ with $\langle a, b \rangle = \sum_{j \in \mathbb{Z}^n} \overline{a_j} \, b_j$

Remark: $\langle a, b \rangle = \overline{\langle b, a \rangle}$ and $\langle a, a \rangle \geqslant 0$ with equality iff $a = \mathbf{0}$