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Prop.

(1) For all $a \in G$, ${}_{G_0}[a] = [a]_{G_0} \iff G_0 \triangleleft G$

(2) $G_0 \triangleleft G \Rightarrow [a]_{G_0}[b]_{G_0} = [ab]_{G_0}$

This makes $\{[a]_{G_0} \mid a \in G\}$ a group.

proof

(1) Let $a \in G$.

If ${}_{G_0}[a] = [a]_{G_0}$, that is, $aG_0 = G_0 a$,

then $aG_0a^{-1} = G_0aa^{-1} = G_0$

Therefore $G_0 \triangleleft G$

If $G_0 \triangleleft G$,

then $aG_0 = aG_0a^{-1}a = G_0 a$ for each $a \in G$

(2) I show that $[a]_{G_0} \cdot [b]_{G_0} := [ab]_{G_0}$ is well-defined (independent of the representative element)

If $[a]_{G_0} = [a']_{G_0}$ and $[b]_{G_0} = [b']_{G_0}$,

then $[ab]_{G_0} = G_0(ab)$

$= (G_0 a) b$ ($\because$ associativity)

$= (aG_0) b$ ($\because$ normal)

$= a(G_0 b)$ $\quad$ ($\because$ associativity)

$= a(G_0 b')$ $\quad$ ($\because [b]_{G_0} = [b']_{G_0}$)

$= (aG_0) b'$ $\quad$ ($\because$ associativity)

$= (G_0 a) b'$ $\quad$ ($\because$ normal)

$= (G_0 a') b'$ $\quad$ ($\because [a]_{G_0} = [a']_{G_0}$)

$= G_0 (a'b')$ $\quad$ ($\because$ associativity)

Thus, this operation

$$\begin{array}{ccc} G/G_0 \times G/G_0 & \longrightarrow & G/G_0 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ ([a]_{G_0}, [b]_{G_0}) & \longmapsto & [ab]_{G_0} \end{array}$$

is well-defined.

Then G/G_0 is a group, whose identity element $[e]_{G_0}$ and inverse of $[a]_{G_0}$ is $[a^{-1}]_{G_0}$.

Indeed:

(Associativity) If $a, b, c \in G$,

$$\begin{aligned} ([a]_{G_0} \cdot [b]_{G_0}) \cdot [c]_{G_0} &= [ab]_{G_0} \cdot [c]_{G_0} \\ &= [(ab)c]_{G_0} \\ &= [a(bc)]_{G_0} \\ &= [a]_{G_0} \cdot [bc]_{G_0}. \end{aligned}$$

$= [a]_{G_0}([b]_{G_0} \cdot [c]_{G_0})$

(Identity element) If $a \in G$.

$$[a]_{G_0}[e]_{G_0} = [ae]_{G_0}$$
$$= [a]_{G_0}$$
$$[e]_{G_0} \cdot [a]_{G_0} = [ea]_{G_0}$$
$$= [a]_{G_0}.$$

(Inverse element) If $a \in G$.

$$[a]_{G_0}[a^{-1}]_{G_0} = [aa^{-1}]_{G_0}$$
$$= [e]_{G_0}$$

$\square$