061601856 Nguyen Song Ha

081531238 Bui Dang Cong

I, From Outer Semi-direct product to Inner Semi-direct product

Let $N; H$ be 2 groups and let $\phi : H \to \mathrm{Aut}(N)$ be a homomorphism.

Then we set the Outer Semi-direct product of N and H as the group:

$G \equiv N \rtimes_\phi H = \{(n;h) \mid n \in N; h \in H\}$ with the operation

$(n_1; h_1)(n_2; h_2) = (n_1 \phi_{h_1}(n_2) ; h_1 h_2)$ and note that $e_G = (e_N; e_H)$

Now if we identify N with G_1 and H with G_2, i.e. we set 2 bijective maps

$$\phi_N : N \to G_1 \qquad ; \qquad \phi_H : H \to G_2$$
$$n \to (n; e_H) \qquad\qquad h \to (e_N; h)$$

Then we obtain 2 subgroups of G and their inner semi-direct product generates the group G i.e. $G_1 \odot G_2 = N \rtimes_\phi H$

Proof | Firstly, it's easy to check that $G_1; G_2$ are subgroups of G

Indeed, $\underset{G_1}{\underset{\cap}{(n_1; e_H)}}\underset{G_1}{\underset{\cap}{(n_2; e_H)}} = (n_1 \phi_{e_H}(n_2) ; e_H) = \underset{G_1}{\underset{\cap}{(n_1 n_2 ; e_H)}}$

(Same for G_2)

⊕ G_1 is normal in G

Indeed, for any $(n_1; h_1) \in G$ one has:

$$\begin{aligned}(n_1; h_1)(n; e_H)(n_1; h_1)^{-1} &= (n_1 \phi_{h_1}(n); h_1)(\phi_{h_1^{-1}}(n_1^{-1}); h_1^{-1}) \\ &= (n_1 \phi_{h_1}(n) \phi_{h_1}(\phi_{h_1^{-1}}(n_1^{-1})) ; h_1 h_1^{-1}) \\ &= (n_1 \phi_{h_1}(n) \phi_{e_H}(n_1^{-1}) ; e_H) \\ &= (n_1 \phi_{h_1}(n) n_1^{-1} ; e_H) \in G_1\end{aligned}$$

Since ϕ_{h_1} is an automorphism in N, there is a one-on-one relation between an element of G_1 i.e. $(n; e_H)$ and $(n_1; h_1)(n; e_H)(n_1; h_1)^{-1}$

Thus $(n_1; h_1) G_1 (n_1; h_1)^{-1} = G_1$ i.e. $G_1 \triangleleft G$

⊕ $G_1 \cap G_2 = \{e_G\}$ which is easy to see

⊕ $\forall (n; e_H) \in G_1 ; \forall (e_N; h) \in G_2$ one has:

$(n; e_H)(e_N; h) = (n \phi_{e_H}(e_N) ; h) = (n; h)$

Which means $G_1 G_2 = G$ i.e. $\forall g \in G, \exists g_1; g_2$ s.t. $g = g_1 g_2$ ($g_i \in G_i$

So we have proved that G_1 and G_2 satisfy 3 conditions for semi-direct product i.e. $G_1 \odot G_2 = G$

II, From Inner semi-direct product to Outer semi-direct product

Let G be a group with 2 subgroups $N; H$ satisfying :

$$\left.\begin{array}{l} 1,\ N \text{ is normal in } G \\ 2,\ N \cap H = \{e_G\} \\ 3,\ NH = G \end{array}\right\} \text{ i.e } \quad G = N \odot H$$

Let $n_1; n_2 \in N$; $h_1; h_2 \in H$ one has:

$$n_1 h_1 n_2 h_2 = (n_1 h_1 n_2 h_1^{-1})(h_1 h_2)$$

Since N is normal, we can set : $\begin{array}{l} \phi_{h_1} : N \to N \\ \qquad n \to h_1 n h_1^{-1} \end{array}$

and it's easy to see that ϕ_{h_1} is an automorphism in N

And so it's natural to set : $\begin{array}{l} \phi : H \to \mathrm{Aut}(N) \\ \quad h \to \phi_h \end{array}$

Firstly, check that ϕ is a homomorphism

Indeed, $[\phi_{h_1} \circ \phi_{h_2}](n) = \phi_{h_1}(\phi_{h_2}(n)) = h_1 (h_2 n h_2^{-1}) h_1^{-1}$

for any $h_1; h_2 \in H$

$$= h_1 h_2\, n\, (h_1 h_2)^{-1}$$

$$= \phi_{h_1 h_2}(n)$$

And so we can construct $N \rtimes_\phi H = \{(n;h) \mid n \in N, h \in H\}$ with the operation $(n_1; h_1)(n_2; h_2) = (n_1 \phi_{h_1}(n_2) ; h_1 h_2)$

Finally, we need to show that G is isomorphic to $N \rtimes_\phi H$

Indeed, let $\begin{array}{l} \varphi : G \equiv NH \to N \rtimes_\phi H \\ \qquad\quad nh \to (n; h) \end{array}$ be a bijective map

$$\text{Clearly, } \varphi(n_1 h_1)\varphi(n_2 h_2) = (n_1; h_1)(n_2; h_2) = (n_1 \phi_{h_1}(n_2) ; h_1 h_2)$$

$$= (n_1 h_1 n_2 h_1^{-1} ; h_1 h_2)$$

$$= \varphi(n_1 h_1 n_2 h_2)$$

So φ is an isomorphism from G to $N \rtimes_\phi H$ i.e. $G \simeq N \rtimes_\phi H$