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Report

- Lemma: Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$. Then
$\underset{n\to\infty}{s\text{-}\lim} f_n = f_\infty \iff \underset{n\to\infty}{w\text{-}\lim} f_n = f_\infty$ and $\lim_{n\to\infty} \|f_n\| = \|f_\infty\|$

Proof: "$\Longrightarrow$"
From Schwarz inequality, for $\forall h_1, h_2 \in \mathcal{H}$, we have $|\langle h_1, h_2\rangle| \le \|h_1\|\cdot\|h_2\|$. Then for $\forall g \in \mathcal{H}$,
$|\langle g, f_n - f_\infty\rangle| \le \|g\|\cdot\|f_n - f_\infty\| \longrightarrow 0$ as $n\to\infty$
so $\underset{n\to\infty}{w\text{-}\lim} f_n = f_\infty$.

From Triangle inequality, we have
$|\|f_n\| - \|f_\infty\|| \le \|f_n - f_\infty\| \longrightarrow 0$ as $n \to \infty$
so $\lim_{n\to\infty} \|f_n\| = \|f_\infty\|$

"$\Longleftarrow$"

$$\|f_n - f_\infty\|^2 = \|f_n\|^2 + \|f_\infty\|^2 - \langle f_n, f_\infty\rangle - \langle f_\infty, f_n\rangle$$
$$= \|f_n\|^2 + \|f_\infty\|^2 - (\langle f_n, f_\infty\rangle - \langle f_\infty, f_\infty\rangle) - (\langle f_\infty, f_n\rangle - \langle f_\infty, f_\infty\rangle) - 2\langle f_\infty, f_\infty\rangle \longrightarrow \|f_\infty\|^2 + \|f_\infty\|^2 - 0 - 0 - 2\|f_\infty\|$$
$= 0$ as $n \to \infty$ by $\underset{n\to\infty}{w\text{-}\lim} f_n = f_\infty$ and $\lim_{n\to\infty}\|f_n\| = \|f_\infty\|$
so $\underset{n\to\infty}{s\text{-}\lim} f_n = f$.

$\square$

- Let $B \in K(\mathcal{H})$.
If $f_n \xrightarrow[w]{n\to\infty} f_\infty$, then $Bf_n \xrightarrow[s]{n\to\infty} Bf_\infty$

Proof: Since each weakly convergent sequence is bounded, let $\varphi_n := f_n - f_\infty$, so $\underset{n\to\infty}{w\text{-}\lim} \varphi_n = 0$, then $\exists$ a real number M s.t. $\|\varphi_n\| \le M$ for all n.

By the definition of compact operator, for some given $\varepsilon>0$, choose a finite rank operator T with the form:

$$Tf = \sum_{k=1}^{N} \langle g_k, f\rangle h_k \quad , \text{ where } \{g_k, h_k\}_{k=1}^{N\in\mathbb{N}} \subset \mathcal{H}, \ N<\infty.$$

s.t. $\|B-T\| < \varepsilon/2M$.

Then $\|B\varphi_n\| \leq \|(B-T)\varphi_n\| + \|T\varphi_n\| \leq \varepsilon/2 + \sum_{k=1}^{N} |\langle g_k, \varphi_n\rangle| \cdot \|h_k\|$.

Since $w\text{-}\lim_{n\to\infty} \varphi_n = 0$, $\exists\ n_0 \in \mathbb{N}$ s.t. $|\langle g_k, \varphi_n\rangle| \cdot \|h_k\| < \varepsilon/2N$ for $\forall\ k=1,\cdots,N$ and all $n>n_0$.

$\therefore \sum_{k=1}^{N} |\langle g_k, \varphi_n\rangle| \cdot \|h_k\| < \frac{\varepsilon}{2}$ for all $n>n_0$

Then $\|B\varphi_n\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $n>n_0$

Then $s\text{-}\lim_{n\to\infty} B\varphi_n = 0$, namely $s\text{-}\lim_{n\to\infty} Bf_n = Bf_\infty$.

$\square$

- Let $B \in \mathcal{K}(\mathcal{H})$, A_n and A belong to $\mathcal{B}(\mathcal{H})$.
  If $A_n \xrightarrow[s]{n\to\infty} A_\infty$, then $A_nB \xrightarrow[u]{n\to\infty} A_\infty B$ and $BA_n^* \xrightarrow[u]{n\to\infty} BA_n^*$

Proof: Let $C_n = A_n - A_\infty$, then $s\text{-}\lim_{n\to\infty} C_n = 0$.

Since C_n is bounded, then $\exists$ a real number M s.t. $\|C_n\| \leq M$ for all n.

Let $\{T_N\}$ be a sequence of finite rank operators s.t. $\|B-T_N\| \longrightarrow 0$ as $N\longrightarrow\infty$. Then

$$\begin{aligned}\|C_nBf\| &\leq M\|(B-T_N)f\| + \|C_nT_Nf\| \\ &\leq M\|B-T_N\|\cdot\|f\| + \sum_{k=1}^{N} |\langle g_k, f\rangle| \cdot \|C_nh_k\| \\ &\leq \Big(M\|B-T_N\| + \sum_{k=1}^{N} \|g_k\|\,\|C_nh_k\|\Big)\cdot\|f\|\end{aligned}$$

For some given $\varepsilon>0$, choose N s.t. $\|B-T_N\| < \varepsilon/2M$, choose n_0 s.t. $\|g_k\|\cdot\|C_nh_k\| < \varepsilon/2N$ for all $n>n_0$ and each $k=1,2,\ldots,N$. Then $\|C_nBf\| \leq \varepsilon\|f\|$, namely, $\|C_nB\| \leq \varepsilon$ for all $n>n_0$. Then $A_nB \xrightarrow[u]{n\to\infty} A_\infty B$.

Notice that $\|BA_n^* - BA_\infty^*\| = \|A_nB^* - A_\infty B^*\|$, $\because B \in \mathcal{K}(\mathcal{H})$ $\therefore B^* \in \mathcal{K}(\mathcal{H})$, then we can use the same procedure to prove $BA_n^* \xrightarrow[u]{n\to\infty} BA_\infty^*$.

$\square$