LIU Shuheng

Let $\{g_j, h_j\}_{j=1}^N \subset \mathcal{H}$, and we set for any $f \in \mathcal{H}$

$$Af := \sum_{j=1}^N \langle g_j, f \rangle h_j$$

$A \in \mathcal{B}(\mathcal{H})$ is called a finite rank operator defined by $\{g_j, h_j\}_{j=1}^N$.

Claim that: Let A^* be the adjoint operator of A, then A^* is the finite rank operator defined by $\{h_j, g_j\}_{j=1}^N$.

To see this, note that for any pair of elements $f, \tilde{f}$ in $\mathcal{H}$,

$$\begin{aligned}\langle A^* f, \tilde{f} \rangle &= \langle f, A\tilde{f} \rangle \\ &= \langle f, \sum_{j=1}^N \langle g_j, \tilde{f} \rangle h_j \rangle \\ &= \sum_{j=1}^N \langle g_j, \tilde{f} \rangle \langle f, h_j \rangle \\ &= \langle \sum_{j=1}^N \langle h_j, f \rangle g_j, \tilde{f} \rangle\end{aligned}$$

and which holds if and only if $A^* f = \sum_{j=1}^N \langle h_j, f \rangle g_j$.

$\square$

To see that A (and therefore, A^*, from what we have shown) is bounded, Note that for any $f \in \mathcal{H}$, $\|Af\|^2 = \langle Af, Af \rangle = \langle \sum_{i=1}^N \langle g_i, f \rangle h_i, \sum_{j=1}^N \langle g_j, f \rangle h_j \rangle$ $= \sum_{i,j} \langle f, g_i \rangle \langle g_j, f \rangle \langle h_i, h_j \rangle =: \sum_{i,j} S_{ij}(f)$, and that $|S_{i,j}(f)| \leq \|f\|^2 \|g_i\| \|g_j\| \, |\langle h_i, h_j \rangle| =: \|f\|^2 C_{i,j}$ where $C_{i,j}$ is constant, we have, therefore

$\|Af\|^2 \leq \sum_{i,j} |S_{i,j}(f)| \leq \max_{i,j} C_{i,j} \cdot N^2$ when $\|f\| = 1$, i.e.,

A is ~~constant~~ bounded (So A^* is also ~~constant~~ bounded.) $\square$