

Generalized Eigen vectors

Consider

$$M \in \mathbb{C}^{n \times n}$$

Then the characteristic polynomial:

$$P(\lambda) \equiv |M - \lambda I|$$

has p roots, $p \leq n$

$$\text{i.e. } \{\lambda_i\}_{i=1, p} \text{ s.t. } P(\lambda_i) = 0$$

If each root has multiplicity k_i

$$\text{s.t. } P(\lambda) = \prod_{i=1}^p (\lambda - \lambda_i)^{k_i}$$

$$\text{with } \sum_i k_i = n$$

$$E_i \equiv \{ \vec{v} \in \mathbb{C}^n \mid M\vec{v} = \lambda_i \vec{v} \}$$

$$\dim(E_i) \leq k_i$$

Example 1; if $\dim(E_i) < k_i$:

$$B = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

$$P(\lambda) = \begin{vmatrix} -2-\lambda & 1 \\ 0 & -2-\lambda \end{vmatrix} = (\lambda+2)^2$$

which has one repeated root $\lambda = -2$ with multiplicity $k_1 = 2$ indeed $1 < 2$ ($p \leq n$)

$$\text{and } P(\lambda) = (\lambda - \lambda_1)^{k_1} = (\lambda + 2)^2$$

The eigen vectors \vec{v}_i^j associated with $\lambda_1 = -2$

$$\text{span Ker}(B - \lambda_1 I) = E_1$$

$$\text{Ker}(B - \lambda_1 I) = \text{Ker} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \dim E_1 = 1 < 2 \quad (\dim E_1 < k_1)$$

Note that B is not diagonalizable, as there are not enough \vec{v}_i^j to diagonalize it $i \in [1, p] \subset \mathbb{Z}$ $j \in [1, \dim E_i]$

However, taking $\vec{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{aligned} (A - \lambda I)^2 \vec{x} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \end{aligned}$$

This motivates the following definition:

Definition:

For a given eigenvalue λ , \vec{u} is a generalized eigenvector of rank r if

$$\begin{aligned} (A - \lambda I)^r \vec{u} &= 0 \\ (A - \lambda I)^{r-1} \vec{u} &\neq 0 \end{aligned}$$

Note that if $(A - \lambda I)^m \vec{u} = 0 \Rightarrow (A - \lambda I)^{r-m-1} (A - \lambda I)^{m+1} \vec{u} = (A - \lambda I)^{r-1} \vec{u} = 0$

Definition:

$\{\vec{w}_i\}_{i=1, \dots, r}$ s.t. $\vec{w}_i = (A - \lambda I)^{r-i} \vec{u}$, \vec{u} as above

$\{\vec{w}_i\}_{i=1, \dots, r}$ form a chain of generalized eigenvectors of length r

with $\vec{w}_r = (A - \lambda I)^{r-r} \vec{u} = \vec{u}$

$$\vec{w}_i = (A - \lambda I)^{r-i} \vec{u} \neq 0$$

$(A - \lambda I) \vec{w}_i = (A - \lambda I)^{r-i+1} \vec{u} = 0$ by definition of \vec{u}

So \vec{w}_i is an eigenvector of λ as $\vec{w}_i \neq 0$

Theorem:

The eigenvectors \vec{w}_i are all linearly independent

Using $(A - \lambda I)^m \vec{u} = 0$ for $\forall m \geq r$, \vec{u} as above

since $(A - \lambda I)^{m-r} \underbrace{(A - \lambda I)^r}_{=0} \vec{u} = 0$ by def of \vec{u}

$$\sum_{i=1}^r a_i \vec{w}_i = 0$$

$$\Rightarrow \sum_{i=1}^r a_i (A - \lambda I)^{r-i} \vec{u} = 0$$

$$\Rightarrow (A - \lambda I)^{r-1} \sum_{i=1}^r a_i (A - \lambda I)^{r-i} \vec{u} = 0$$

$$\Rightarrow \sum_{i=1}^r a_i (A - \lambda I)^{2r-i-1} \vec{u} = 0$$

but $(A - \lambda I) \vec{u} = 0$ for $2r-i-1 > r$
 $\Rightarrow r-1 \geq i$
 $\Rightarrow i < r$

so only $(A - \lambda I) \vec{u} \neq 0$

i.e. $a_r (A - \lambda I) \vec{u} = 0$ but $(A - \lambda I) \vec{u} \neq 0$
 $\Rightarrow a_r = 0$

knowing that $a_r = 0$

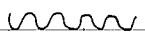
we have: $\sum_{i=1}^{r-1} a_i (A - \lambda I) \vec{u} = 0$

now applying $(A - \lambda I)^{r-2}$ we get $a_{r-1} = 0$

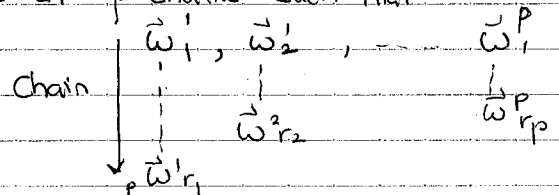
continuing in a similar inductive fashion we get $a_i = 0$ $i \in \{1, 2, \dots, r\}$

$\Rightarrow \vec{w}_i$ are linearly independent

Theorem:



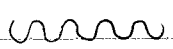
For an eigenvalue λ of multiplicity k $i \in \{1, 2, \dots, p\}$
 there exist p chains such that: $p = \dim(E(\lambda))$



we have $\sum_{i=1}^p r_i = k$

" From our example we had $\vec{w}_1^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ "

$\vec{w}_2^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
 $\sum_{i=1}^1 r_i = 2 = k$



Theorem:

$\{\vec{w}_i^j\}_{i=1, r_p}^{j=1, p}$ is a set of linearly independent vectors if and only if $\{\vec{w}_i^j\}_{j=1, p}$ are independent.

Using the chains as basis we can write

$M \in \mathbb{C}^{n \times n}$ as :

$$M' = \begin{pmatrix} \lambda_1 & 1 & & 0 \\ & \lambda_1 & 1 & \\ & & \ddots & \ddots \\ & & & \lambda_1 \\ & & & & \lambda_2 \\ & & & & & \ddots \\ & & & & & & \lambda_r \end{pmatrix}$$

$$\text{as } (M - \lambda I) \vec{w}_i = (M - \lambda I) \vec{w}_{i-1} \quad \vec{w}_i = \vec{w}_{i-1} \quad \text{by definition of } \vec{w}_i$$

$$\Rightarrow M \vec{w}_i = \lambda \vec{w}_i + \vec{w}_{i-1}$$

$$\text{with } M' = P^{-1} M P \quad P = \begin{pmatrix} \vec{w}_1 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_r \\ & & & & \vec{w}_r \end{pmatrix}$$

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References :

- 1) Lecture notes on Generalized Eigenvectors for Systems with repeated Eigenvalues, University of Oslo website
- 2) Jordan Normal Form, Wikipedia article
- 3) Generalized Eigenvectors, University of Pennsylvania