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Generalized Eigenvectors

Consider

$M \in \mathbb{C}^{n \times n}$

Then the charecteristic polynomial:

$P(\lambda) \equiv |M - \lambda I|$

has p roots, $p \leq n$

i.e $\{\lambda_i\}_{i=1,p}$ s.t. $P(\lambda_i) = 0$

If each root has multiplicity k_i

s.t $P(\lambda) = \prod_{i=1}^{p} (\lambda - \lambda_i)^{k_i}$

with $\sum_i^p k_i = n$

$E_i \equiv \{ \vec{v} \in \mathbb{C}^n \mid M\vec{v} = \lambda_i \vec{v} \}$

$\dim(E_i) \leq k_i$

Example 1; of $\dim(E_i) < k_i$:

$B = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$

$P(\lambda) = \left| \begin{pmatrix} -2-\lambda & 1 \\ 0 & -2-\lambda \end{pmatrix} \right| = (2+\lambda)^2$

which has one repeated root $\lambda_1 = -2$ with multiplicity $k_1 = 2$

indeed $1 < 2$ $(p \leq n)$

and $P(\lambda) = (\lambda - \lambda_1)^{k_1} = (\lambda + 2)^2$

The eigenvectors $\vec{v}_1^j$ associated with $\lambda_1 = -2$

span $\ker(B - \lambda_1 I) = E_1$

$\ker(B - \lambda_1 I) = \ker \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow \dim E_1 = 1 < 2$ $(\dim E_1 < k_1)$

Note that B is not diagonalizable, as there are not enough $\vec{v}_i^j$ to diagonalize it $i \in [1, p] \subset \mathbb{Z}$ $j \in [1, \dim E_i] \cap \mathbb{Z}$

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However, taking $\vec{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$(A - \lambda_1 I)^2 \vec{u} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

This motivates the following definition:

Definition:

For a given eigenvalue λ_i, $\vec{u}$ is a generalized eigenvector of rank r if

$$(A - \lambda I)^r \vec{u} = 0$$
$$(A - \lambda I)^{r-1} \vec{u} \neq 0$$

Note that if $(A - \lambda I)^m \vec{u} = 0$, $m \in \mathbb{Z}^+/0$ $\Rightarrow$ $(A - \lambda I)^{r-m-1} (A - \lambda I)^m \vec{u} = (A - \lambda I)^{r-1} \vec{u} = 0$

Definition:

$\{\vec{\omega}_i\}_{i=1,r}$ s.t. $\vec{\omega}_i \equiv (A - \lambda)^{r-i} \vec{u}$, $\vec{u}$ as above

$\{\vec{\omega}_i\}_{i=1,r}$ form a chain of generalized eigenvectors of lenth r

with $\vec{\omega}_r = (A - \lambda I)^{r-r} \vec{u} = \vec{u}$

$\vec{\omega}_1 = (A - \lambda I)^{r-1} \vec{u} \neq 0$

$(A - \lambda I) \vec{\omega}_1 = (A - \lambda I)^r \vec{u} = 0$ by definition of $\vec{u}$

So $\vec{\omega}_1$ is an eigenvector of λ as $\vec{\omega}_1 \neq 0$

Theorem:

The eigenvectors $\vec{\omega}_i$ are all linearly independent

Using $(A - \lambda I)^m \vec{u} = 0$ for $\forall m \geq r$, $\vec{u}$ as above

since $(A - \lambda I)^{m-r} \underbrace{(A - \lambda I)^r \vec{u}}_{0} = 0$ bey def of $\vec{u}$

$$\sum_{i=1}^{r} a_i \vec{\omega}_i = 0$$

$$\Rightarrow \sum_{i=1}^{r} a_i (A - \lambda I)^{r-i} \vec{u} = 0$$

$$\Rightarrow (A - \lambda I)^{r-1} \sum_{i=1}^{r} a_i (A - \lambda I)^{r-i} \vec{u} = 0$$

$$\Rightarrow \sum_{i=1}^{r} a_i (A - \lambda I)^{2r-i-1} \vec{u} = 0$$

but $(A-\lambda I)^{2r-i-1}\vec{u} = 0$ for $2r-i-1 \geq r$

$\Rightarrow r-1 \geq i$

$\Rightarrow i < r$

so only $(A-\lambda I)^{2r-r-1}\vec{u} \neq 0$

i,e $a_r (A-\lambda I)^{r-1}\vec{u} = 0$ but $(A-\lambda I)^{r-1}\vec{u} \neq 0$

$\Rightarrow a_r = 0$

knowing that $a_r = 0$

we have :

$$\sum_{i=1}^{r-1} a_i (A-\lambda I)^{r-1}\vec{u} = 0$$

now applying $(A-\lambda I)^{r-2}$ we get $a_{r-1} = 0$

continuing in a similar inductive fashion we get

$a_i = 0 \quad i \in [1, 2, \ldots r]$

$\Rightarrow \vec{\omega}_i$ are linearly independent

Theorem :

For an eigenvalue λ_i of multiplicity k $\quad i \in [1, 2, \ldots P]$

there exist p chains such that : $\quad p = \dim(E(\lambda))$

$$\begin{array}{llll} \vec{\omega}^1_1, & \vec{\omega}^2_1, & \ldots & \vec{\omega}^p_1 \\ \vdots & \vdots & & \vdots \\ \vdots & \vec{\omega}^2_{r_2} & & \vec{\omega}^p_{r_p} \\ \vec{\omega}^1_{r_1} & & & \end{array}$$

Chain $\downarrow$

we have $\displaystyle\sum_{i=1}^{p} r_i = k$

"From our example we had $\vec{\omega}^1_1 = \begin{pmatrix}1\\0\end{pmatrix}$
$\omega^1_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$$\sum_{i=1}^{1} r_i = 2 = k$$ "

Theorem :

$\{\vec{\omega}^j_i\}^{j=1,p}_{i=1,r_p}$ is a set of linearly independent vectors if and only if $\{\vec{\omega}^j_1\}^{j=1,p}$ are independent.

Using the chains as basis we can write $M \in \mathbb{C}^{n\times n}$ as:

$$M' = \begin{pmatrix} \begin{matrix}\lambda_1 & 1 & & 0\\ & \lambda_1 & 1 & \\ & 0 & \ddots & 1 \\ & & & \lambda_1\end{matrix} & & \\ & \ddots & \\ & & \begin{matrix}\lambda_q & & 0 \\ & \ddots & 1 \\ & & \lambda_q\end{matrix}\end{pmatrix}$$

as $(M-\lambda I)\vec{\omega}_i = (M-\lambda I)^{r-(i+1)}\vec{u} = \vec{\omega}_{i-1}$ $\quad i<r \quad$ by definition of $\vec{\omega}_i$

$\Rightarrow M\vec{\omega}_i = \lambda\vec{\omega}_i + \vec{\omega}_{i-1}$

with $M' = P^{-1} M P \qquad P = \left(\vec{\omega}^{1}_{1,\lambda_1} - - - - - - \vec{\omega}^{p}_{r_p,\lambda_q}\right)$

References :

1) Lecture notes on Generalized Eigenvectors for Systems with repeated Eigenvalues, University of Oslo website

2) Jordan Normal Form, Wikipedia article

3) Generalized Eigenvectors, University of Pennsylvania