
Homework 7

Exercise 1 Let us consider $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $f(x, y, z) = e^{xy} \cos(z)$ for any $(x, y, z) \in \mathbb{R}^3$. Assume also that $x = tu$, $y = \sin(tu)$ and $z = u^2$ for some $t, u \in \mathbb{R}$. By setting

$$F(t, u) := f(tu, \sin(tu), u^2)$$

Compute the derivative $\partial_2 F \equiv \partial_u F$ by three different methods: once by a direct computation, once as one component of the derivative of the composition of two functions (chain rule), and once with the formula often seen in the literature

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u}.$$

Can you explain where this formula comes from ?

Exercise 2 (Spherical coordinates) Consider the map $\Phi : [0, \infty) \times [0, 2\pi) \times [0, \pi) \rightarrow \mathbb{R}^3$ with

$$\Phi(r, \theta, \varphi) := (r \cos(\theta) \sin(\varphi), r \sin(\theta) \sin(\varphi), r \cos(\varphi)).$$

Compute the Jacobian matrix corresponding to this function.

Recall that a *vector field* is a function f defined on an open set $\Omega \subset \mathbb{R}^n$ and taking values in \mathbb{R}^n . For such a vector field f , if there exists a differentiable function $\phi : \Omega \rightarrow \mathbb{R}$ such that $f = \nabla \phi$ we say that ϕ is a *potential function* for f .

Exercise 3 For the following functions, does it exist a potential function ?

(i) $f(x, y) = (y, x)$,

(ii) $f(x, y) = (3x^2y + 2x + y^3, x^3 + 3xy^2 - 2y)$,

(iii) $f(x, y) = (\cos(x), \sin(y))$,

(iv) $f(x, y, z) = (x^2 - yz, y^2 - zx, z^2 - xy)$.

Exercise 4 a) Let $\Omega \subset \mathbb{R}^2$ be open and let $F : \Omega \rightarrow \mathbb{R}^2$ be of class C^1 on Ω . Let us also set $F = (f_1, f_2)$. Show that if

$$\partial_x f_2 \neq \partial_y f_1$$

then F does not admit a potential function of class C^2 .

b) What would be a similar statement for a function $F : \Omega \rightarrow \mathbb{R}^3$ if Ω is an open subset of \mathbb{R}^3 .

c) What about the n -dimensional case, and how many conditions have to be satisfied ?