
Homework 6

Exercise 1 Draw the unit ball of \mathbb{R}^2 with the three norms $\|\cdot\|_2$, $\|\cdot\|_1$ and $\|\cdot\|_\infty$.

Exercise 2 Let us consider $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ sufficiently differentiable on \mathbb{R}^3 . One also sets $F = {}^T(F_1, F_2, F_3)$ and

$$\begin{aligned}\operatorname{grad}(f) &\equiv \nabla f = {}^T(\partial_1 f, \partial_2 f, \partial_3 f) \\ \operatorname{div}(F) &\equiv \nabla \cdot F = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3 \\ \operatorname{curl}(F) &\equiv \nabla \times F = {}^T(\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1) \\ \Delta f &= \partial_1^2 f + \partial_2^2 f + \partial_3^2 f.\end{aligned}$$

Show then the following relations:

- (i) $\operatorname{div}(fF) = f \operatorname{div}(F) + \operatorname{grad}(f) \cdot F$,
- (ii) $\operatorname{div}(\operatorname{curl}(F)) = 0$,
- (iii) $\operatorname{curl}(\operatorname{grad}(f)) = 0$,
- (iv) $\operatorname{curl}(\operatorname{curl}(F)) = \operatorname{grad}(\operatorname{div}(F)) - \Delta F$, with $\Delta F = {}^T(\Delta F_1, \Delta F_2, \Delta F_3)$.

In the following exercises we call a *vector field* a function f defined on an open set $\Omega \subset \mathbb{R}^n$ and taking values in \mathbb{R}^n . In other words, for $\Omega \subset \mathbb{R}^n$ open, a function $f : \Omega \rightarrow \mathbb{R}^d$ is a vector field if $d = n$.

Exercise 3 Provide a picture for the following vector fields:

- (i) $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $f(x, y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,
- (ii) $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $g(x, y) = x E_1 + y E_2$,
- (iii) $\nabla k : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ for $k : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $k(x, y) = \frac{1}{1+x^2+y^2}$.

Exercise 4 Consider the functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}^2$ defined by $\varphi(t) := \begin{pmatrix} \cos(t) \\ t^2 \end{pmatrix}$, and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x, y) := e^{3x+2y}$. We consider the composition of these two functions, namely $F : \mathbb{R} \rightarrow \mathbb{R}$ given by $F = f \circ \varphi$. Compute the derivative of this function by two different methods: once by a direct computation, and once as the derivative of a composed function (chain rule).