Chapter 3

The Dixmier trace

The aim of this chapter is to present the construction of Dixmier of a non-normal tracial weight on $\mathscr{B}(\mathcal{H})$. Even if the paper of Dixmier [Dix] is only 2 pages long, we will use more pages for understanding and explaining the details. One reason for devoting so much time for this construction is that this trace had an enormous impact on the program of A. Connes in non-commutative geometry, and also several interesting applications. Such developments will be presented in the following chapters.

Before starting with the construction, let us just mention another non-trivial (but non-interesting) non-normal tracial weight on $\mathscr{B}(\mathcal{H})$. For any $B \in \mathscr{B}(\mathcal{H})_+$ we set

$$\tau(B) := \begin{cases} \mathrm{Tr}(B) & \text{if } B \in \mathscr{F}(\mathcal{H}) \\ \infty & \text{if } B \notin \mathscr{F}(\mathcal{H}). \end{cases}$$

Note that the Dixmier trace will not be of this form. One of its special features is to vanish on the usual trace class elements of $\mathscr{B}(\mathcal{H})$.

3.1 Invariant states

The construction of the Dixmier trace relies on an invariant state on $\ell_\infty \equiv \ell_\infty(\mathbb{N})$. We provide now some information on such a state, following closely the paper [CS1] to which we refer for part of the proofs.

A *state* on ℓ_∞ consists in a positive linear functional $\omega : \ell_\infty \to \mathbb{C}$ satisfying $\omega(\mathbf{1}) = 1$. Here we use the notation $\mathbf{1}$ for the element $(1, 1, 1, \dots) \in \ell_\infty$, and recall that the last condition implies that $\|\omega\| = 1$, see [Mur, Corol. 3.3.5]. Clearly, $\|\omega\|$ denotes the norm of ω as an element of $\ell_\infty(\mathbb{N})^*$. We also recall that positivity means that $\omega(a) \geq 0$ for any $a = (a_n) \in \ell_\infty$ satisfying $a_n \geq 0$ for any $n \in \mathbb{N}$. The set of all states on ℓ_∞ is denoted by $\mathcal{S}(\ell_\infty)$.

By the positivity of ω and its normalization, let us already observe that for any real-valued $a \in \ell_\infty$ one has

$$\inf_n a_n \leq \omega(a) \leq \sup_n a_n. \tag{3.1}$$

We refer to [Lor] for a general introduction on states on ℓ_∞.

Let us now introduce three operations on ℓ_∞, namely the shift operator $S:\ell_\infty\to\ell_\infty$, the Cesàro operator $H:\ell_\infty\to\ell_\infty$, and the dilation operators $D_n:\ell_\infty\to\ell_\infty$ for any $n\in\mathbb{N}$ defined by

$$\begin{aligned} S\big((a_1,a_2,a_3,\dots)\big) &= (a_2,a_3,a_4,\dots),\\ H\big((a_1,a_2,a_3,\dots)\big) &= \Big(a_1,\frac{a_1+a_2}{2},\frac{a_1+a_2+a_3}{3},\dots\Big),\\ D_n\big((a_1,a_2,a_3,\dots)\big) &= \big(\underbrace{a_1,\dots,a_1}_{n},\underbrace{a_2,\dots,a_2}_{n},\dots\big). \end{aligned}$$

The following properties of these operations can easily be checked:

Exercise 3.1.1. *The three operators $S,H,D_n:\ell_\infty\to\ell_\infty$ leave the positive cone $(\ell_\infty)_+$ invariant, leave $\mathbf{1}$ invariant and have norm 1. In addition, $\{D_n\}$ is an Abelian semigroup.*

More interesting relations can also be shown:

Extension 3.1.2. *The following properties hold:*

(i) $D_nS=S^dD_n$ *for any* $n\in\mathbb{N}$,

(ii) $\big(HS-SH\big)(a)\in c_0$ *for any* $a\in\ell_\infty$,

(iii) $\big(HD_n-D_nH\big)(a)\in c_0$ *for any* $a\in\ell_\infty$.

The shift operator allows us to introduce an important concept on $\mathcal{S}(\ell_\infty)$: A state ω on ℓ_∞ is called *a Banach limit* if it is invariant under translations, namely if $\omega(Sa)=\omega(a)$ for any $a\in\ell_\infty$. As a consequence of this property a Banach limit always satisfies $\omega(a)=0$ if $a\in c_0$. The set all Banach limits will be denoted by $\mathcal{BL}(\ell_\infty)$. Note that for Banach limits the inequalities (3.1) can be slightly improved, namely

$$\liminf_{n\to\infty} a_n \le \omega(a) \le \limsup_{n\to\infty} a_n. \tag{3.2}$$

Subsequently we shall prove the existence of invariant states. The main argument in the proof is the Markov-Kakutani fixed point theorem, that we first recall.

Theorem 3.1.3 (Markov-Kakutani)**.** *Let $\mathcal{M}$ be a locally convex Hausdorff space and let Ω be a non-empty compact and convex subset of $\mathcal{M}$. Let $\mathscr{F}$ be an Abelian semigroup of continuous linear operators on $\mathcal{M}$ which satisfies $F(\Omega)\subset\Omega$ for any $F\in\mathscr{F}$. Then there exists an element $x\in\Omega$ such that $F(x)=x$ for all $F\in\mathscr{F}$.*

We shall now use this theorem for the space $\big(\ell_\infty(\mathbb{N})\big)^*$ endowed with the weak*-topology. The following statement and proof is borrowed from [CS1, Thm. 4.3].

Theorem 3.1.4. *There exists a state $\tilde\omega$ on ℓ_∞ such that for all $n\ge1$ one has*

$$\tilde\omega\circ S=\tilde\omega\circ H=\tilde\omega\circ D_n=\tilde\omega.$$

In the following proof, we shall use the convenient notations

$$S^*\omega := \omega \circ S, \quad H^*\omega = \omega \circ H, \quad D_n^*\omega = \omega \circ D_n \qquad \forall \omega \in \mathcal{S}(\ell_\infty).$$

Proof. Let us set $\Omega_0 := \mathcal{BL}(\ell_\infty)$, which is the set of Banach limits, and observe that it is convex and weak*-compact, by Banach-Alaoglu theorem. Let us also observe that $D_n^*(\Omega_0) \subset \Omega_0$. Indeed by the content of the Extension 3.1.2 one infers that $S^*D_n^* = D_n^*(S^*)^d$, and therefore for any $\omega \in \Omega_0$

$$S^*(D_n^*\omega) = D_n^*(S^*)^d\omega = D_n^*\omega$$

which implies that $D_n^*\omega$ belongs to Ω_0, bt its definition. As a consequence, one can apply Theorem 3.1.3 to the set Ω_0 and to the Abelian semi-group $\{D_n^*\}$. The resulting set of fixed points will be denoted by Ω_1, namely

$$\Omega_1 := \big\{\omega \in \mathcal{S}(\ell_\infty) \mid S^*\omega = \omega \text{ and } D_n^*\omega = \omega \ \forall n \in \mathbb{N}\big\}.$$

This set is non-empty, and again it is convex and weak*-compact.

Let us now show that $H^*(\Omega_1) \subset \Omega_1$. Recall that for any $\omega \in \mathcal{BL}(\ell_\infty)$ and any $a \in c_0$ one has $\omega(a) = 0$. One then infers again from Extension 3.1.2 that for $\omega \in \Omega_1$ and any $a \in \ell_\infty$ one has

$$\big(D_n^*H^*\omega - H^*D_n^*\omega\big)(a) = \omega\Big(\big(HD_n - D_nH\big)(a)\Big) = 0.$$

As a consequence it follows that $D_n^*H^*\omega = H^*D_n^*\omega = H^*\omega$. Similarly, one also gets from Extension 3.1.2 that $S^*H^*\omega = H^*\omega$ for any $\omega \in \Omega_1$. These two properties imply that $H^*\omega$ belong to Ω_1, or equivalently $H^*(\Omega_1) \subset \Omega_1$. By applying once again Theorem 3.1.3 to the set Ω_1 and to the semi-group $\{(H^*)^d\}$ we conclude that there exists $\tilde{\omega} \in \Omega_1$ such that $H^*\tilde{\omega} = \tilde{\omega}$. Such a state $\tilde{\omega}$ satisfies all the requirements of the statement. □

3.2 Additional sequence spaces

Let us still introduce some additional sequence spaces which complement the ones already introduced in Examples 2.3.10. These spaces were not mentioned in the paper [Dix] but one of them will appear naturally in this context. Note that in Chapter 2 we concentrated on normed ideals. However, the Calkin correspondence in Theorem 2.4.5 is much stronger since it does not require to speak about norms. Here we take advantage of this fact.

First of all, for any $p \geq 1$ recall that

$$\ell_{p,w} = \big\{a \in c_0 \mid a_n^* \in O\big(n^{-1/p}\big)\big\}.$$

This clearly defines a Calkin space, see Definition 2.4.4. The corresponding two-sided ideals of $\mathscr{B}(\mathcal{H})$ is denoted by $\mathscr{J}_{p,w}$. Note that these spaces are also often denoted by $\ell_{p,\infty}$ and $\mathscr{L}_{p,\infty}$, and one has

$$\mathscr{L}_{p,\infty} = \big\{A \in \mathscr{K}(\mathcal{H}) \mid \mu_n(A) \in O\big(n^{-1/p}\big)\big\}. \tag{3.3}$$

For applications, the space $\mathscr{L}_{1,\infty}$ is the most important one of the above family. Note that one can define a quasi-norm[1] on this space by the formula

$$\|A\|_{1,\infty} := \sup_{n\geq 1} n\,\mu_n(A).$$

For an increasing and concave function $\psi : [0,\infty) \to [0,\infty)$ satisfying $\lim_{x\searrow 0}\psi(x) = 0$ and $\lim_{x\to\infty}\psi(x) = \infty$ we define *the Lorentz sequence space*

$$m_\psi := \Big\{a \in c_0 \mid \|a\|_{m_\psi} := \sup_{n\geq 1}\frac{1}{\psi(n)}\sum_{j=1}^{d} a_j^* < \infty\Big\}. \tag{3.4}$$

Examples of such functions ψ are $x \mapsto x^\alpha$ or $x \mapsto \big(\ln(x+1)\big)^\alpha$ for any $\alpha \in (0,1]$. Again, m_ψ is a Calkin space, and the corresponding two-sided ideal is denoted by $\mathscr{J}_\psi$. Note that in the special case $\psi(x) = \ln(x+1)$ the notations $m_{1,\infty}$ and $\mathscr{M}_{1,\infty}$ are also often used in the literature, and one has

$$\mathscr{M}_{1,\infty} = \Big\{A \in \mathscr{K}(\mathcal{H}) \mid \sup_{n\geq 1}\frac{1}{\ln(n+1)}\sum_{j=1}^{d}\mu_j(A) < \infty\Big\}. \tag{3.5}$$

Remark 3.2.1. *The notations in the literature are not fully fixed and one has to pay attention to the definition used in each paper or book. The spaces $\mathscr{L}_{1,\infty}$ and $\mathscr{M}_{1,\infty}$ are often presented with different notations. We refer also to the Example 1.2.9 in [LSZ].*

Exercise 3.2.2. *Show that the following inclusions hold: $\ell_1 \subset \ell_{1,\infty} \subset m_{1,\infty}$. For that purpose one can also look at [LSZ, Lem. 1.2.8].*

3.3 Dixmier's construction

Even if the following proof does not correspond exactly to the content of [Dix] it is very close to it. For the arguments we mainly follow [Les, Sec. 2.3] and [CS1, Sec. 5.1].

Theorem 3.3.1. *Let ω be a state on ℓ_∞ which vanishes on c_0 and which is invariant under D_2. For any $A \in \big(\mathscr{M}_{1,\infty}\big)_+$ let us set*

$$\mathrm{Tr}_\omega(A) := \omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{d}\mu_j(A)\big)_{n\in\mathbb{N}}\Big). \tag{3.6}$$

Then Tr_ω extends by linearity to a non-trivial trace on $\mathscr{M}_{1,\infty}$, and by setting $\mathrm{Tr}_\omega(A) = \infty$ for all $A \in \mathscr{B}(\mathcal{H})_+ \setminus \big(\mathscr{M}_{1,\infty}\big)_+$ one extends Tr_ω to a non-normal tracial weight on $\mathscr{B}(\mathcal{H})$. If $A \in \mathscr{J}_1$, then $\mathrm{Tr}_\omega(A) = 0$.

[1] A quasi-norm Φ on a complex vector space V is a map $V \to \mathbb{R}_+$ which satisfies for any V and $\lambda \in \mathbb{C}$ the following properties: i) $\Phi(\lambda a) = |\lambda|\Phi(a)$, ii) $\Phi(a+b) \leq c\big(\Phi(a)+\Phi(b)\big)$ for some $c > 0$, iii) $\Phi(a) = 0$ if and only if $a = 0$.

Before starting with the proof, let us mention that the existence of a state on ℓ_∞ satisfying the condition required by this theorem is already a consequence of Theorem 3.1.4. In fact, the states mentioned in this theorem satisfy an unnecessary condition related to the Cesàro operator H. The larger subset Ω_1 of states mentioned in the proof of Theorem 3.1.4 is also suitable for our purpose. Let us also mention that alternative notations are often used for (3.6) as for example

$$\mathrm{Tr}_\omega(A) \equiv \omega - \lim_{n\to\infty} \frac{1}{\ln(n+1)} \sum_{j=1}^{d} \mu_j(A) \equiv \lim_{\omega} \frac{1}{\ln(n+1)} \sum_{j=1}^{d} \mu_j(A). \tag{3.7}$$

One reason for these notations is that if the sequence $\big(\frac{1}{\ln(n+1)} \sum_{j=1}^{d} \mu_j(A)\big)_{n\in\mathbb{N}}$ has a limit, then one has $\mathrm{Tr}_w(A) = \lim_{n\to\infty} \frac{1}{\ln(n+1)} \sum_{j=1}^{d} \mu_j(A)$. This property clearly follows from the facts that $\omega(\mathbf{1}) = 1$ and that $\omega(a) = 0$ for any $a \in c_0$.

The proof of the above statement is divided into several lemmas and exercises. For each of them, the assumptions of Theorem 3.3.1 are implicitly taken into account. First of all, recall that the notions of positive homogeneous and additive have been introduced just before Definition 2.6.7.

Lemma 3.3.2. Tr_ω *is positive homogeneous and additive on* $\big(\mathscr{M}_{1,\infty}\big)_+$.

Proof. Homogeneity property directly follows from the property $\mu_n(\lambda A) = \lambda \mu_n(A)$ for any $\lambda \geq 0$. The proof of the additivity is much more difficult and will use the assumptions made on the state ω.

i) For shortness let us set

$$\sigma_n(A) := \sum_{j=1}^{d} \mu_j(A) \quad \text{for any } A \in \mathscr{K}(\mathcal{H})_+, \tag{3.8}$$

and observe that for any $A, B \in \mathscr{K}(\mathcal{H})$ and any $n \in \mathbb{N}$ the following inequalities hold:

$$\sigma_n(A+B) \leq \sigma_n(A) + \sigma_n(B) \leq \sigma_{2n}(A+B). \tag{3.9}$$

Their proof is quite similar to the min-max principle introduced in Theorem 2.2.1. Indeed, one easily observes that

$$\sigma_n(A) = \sup\big\{\mathrm{Tr}(AP) \mid P \in \mathscr{P}(\mathcal{H}) \text{ with } \dim(P\mathcal{H}) = n\big\}.$$

The first inequality follows then directly from this observation and from the linearity of the trace Tr. For the second, fixed any $\varepsilon > 0$ and let P_A, P_B be such that $\dim(P_A\mathcal{H}) = n = \dim(P_B\mathcal{H})$ and $\mathrm{Tr}(AP_A) > \sigma_n(A) - \varepsilon$ and $\mathrm{Tr}(BP_B) > \sigma_n(B) - \varepsilon$. By setting P for the orthogonal projection on $P_A\mathcal{H} + P_B\mathcal{H}$ (often denoted by $P := P_A \vee P_B$) then we infer that

$$\mathrm{Tr}\big((A+B)P\big) = \mathrm{Tr}(AP) + \mathrm{Tr}(BP) \geq \mathrm{Tr}(AP_A) + \mathrm{Tr}(BP_B) > \sigma_n(A) + \sigma_n(B) - 2\varepsilon.$$

Since $\dim(P\mathcal{H}) \leq 2n$ and since ε is arbitrarily small, one gets

$$\sigma_{2n}(A+B) \geq \mathrm{Tr}\big((A+B)P\big) \geq \sigma_n(A) + \sigma_n(B)$$

which corresponds to the second inequality of (3.9).

ii) Let us now define $a, b, c \in \ell_\infty$ by

$$a_n := \frac{1}{\ln(n+1)}\sigma_n(A),\ b_n := \frac{1}{\ln(n+1)}\sigma_n(B),\ \text{and } c_n := \frac{1}{\ln(n+1)}\sigma_n(A+B).$$

Then the inequality (3.9) reads

$$c_n \leq a_n + b_n \leq \tfrac{\ln(2n+1)}{\ln(n+1)}\, c_{2n}. \tag{3.10}$$

The first inequality together with the positivity of the state ω directly leads to the inequality $\mathrm{Tr}_\omega(A+B) \leq \mathrm{Tr}_\omega(A) + \mathrm{Tr}_\omega(B)$ for any $A, B \in \big(\mathscr{M}_{1,\infty}\big)_+$. On the other hand, since $\lim_{n\to\infty} \frac{\ln(2n+1)}{\ln(n+1)} = 1$ and since ω vanishes on c_0 we infer that

$$\omega\big((c_{2n})_{n\in\mathbb{N}}\big) = \omega\Big(\big(\tfrac{\ln(2n+1)}{\ln(n+1)} c_{2n}\big)_{n\in\mathbb{N}}\Big).$$

Thus, since we will show below that $\omega\big((c_{2n})_{n\in\mathbb{N}}\big) = \omega\big((c_n)_{n\in\mathbb{N}}\big)$, one infers from (3.10) that $\omega(a) + \omega(b) \leq \omega(c)$, or in other words that $\mathrm{Tr}_\omega(A) + \mathrm{Tr}_\omega(B) \leq \mathrm{Tr}_\omega(A+B)$. The two inequalities obtained above prove the additivity of the map Tr_ω.

iii) It remains to show that

$$\omega\big((c_{2n})_{n\in\mathbb{N}}\big) = \omega\big((c_n)_{n\in\mathbb{N}}\big). \tag{3.11}$$

For that purpose, let us simply write the l.h.s. by $\omega\big((c_{2n})\big)$, and observe that by the invariance of ω under D_2 one has

$$\omega\big((c_{2n})\big) = \omega\big(D_2(c_{2n})\big) = \omega\big((c_2, c_2, c_4, c_4, c_6, c_6, \dots)\big).$$

Then, since $\omega(a) = 0$ for any $a \in c_0$, it is sufficient to show that

$$(c_2, c_2, c_4, c_4, c_6, c_6, \dots) - (c_1, c_2, c_3, c_4, c_5, c_6, \dots) \in c_0$$

in order to obtain (3.11). Thus, we are left in proving that $\lim_{n\to\infty}\big(c_{2n} - c_{2n-1}\big) = 0$.

By the definitions of quantities introduced so far one has

$$\begin{aligned} & c_{2n} - c_{2n-1} \\ &= \frac{1}{\ln(2n+1)}\sigma_{2n}(A+B) - \frac{1}{\ln(2n)}\sigma_{2n-1}(A+B) \\ &= \big(\frac{1}{\ln(2n+1)} - \frac{1}{\ln(2n)}\big)\sigma_{2n-1}(A+B) + \frac{1}{\ln(2n+1)}\mu_{2n}(A+B). \end{aligned}$$

Clearly, the second term on the last line tends to 0 as $n \to \infty$. For the first term of the last line, since $A, B \in \big(\mathscr{M}_{1,\infty}\big)_+$, one infers that $\sigma_{2n-1}(A+B) = O\big(\ln(2n)\big)$. Then, since $\big(\frac{1}{\ln(2n+1)} - \frac{1}{\ln(2n)}\big) = o\big(\frac{1}{\ln(2n+1)}\big)$ one deduces that the first term goes to 0 as $n \to \infty$ as well. This completes the proof of the statement. $\square$

Before extending the map Tr_ω, let us observe that this map is non-trivial.

Exercise 3.3.3. *Show that there exists an element $A \in \big(\mathscr{M}_{1,\infty}\big)_+$ which satisfies $\mathrm{Tr}_\omega(A) = 1$.*

By linearity, the map Tr_ω can then be extended to any element of $\mathscr{M}_{1,\infty}$. More precisely, for any self-adjoint $B \in \mathscr{M}_{1,\infty}$ we set $\mathrm{Tr}_\omega(B) = \mathrm{Tr}_\omega(B_+) - \mathrm{Tr}_\omega(B_-)$, and the Dixmier trace for an arbitrary $B \in \mathscr{M}_{1,\infty}$ is defined by $\mathrm{Tr}_\omega(B) = \mathrm{Tr}_\omega\big(\Re(B)\big) + i\mathrm{Tr}_\omega\big(\Im(B)\big)$. In addition, by setting $\mathrm{Tr}_\omega(A) = \infty$ for all $A \in \mathscr{B}(\mathcal{H})_+ \setminus \big(\mathscr{M}_{1,\infty}\big)_+$ one gets that Tr_ω is a weight on $\mathscr{B}(\mathcal{H})$. Also, since $\mu_j(BB^*) = \mu_j(B^*B)$ for any $B \in \mathscr{K}(\mathcal{H})$ and when these expressions are different from 0 one easily infers that Tr_ω is a tracial weight on $\mathscr{B}(\mathcal{H})$

Exercise 3.3.4. *Show that for any $A \in \mathscr{M}_{1,\infty}$ one has $|\mathrm{Tr}_\omega(A)| \leq \|A\|_{\mathscr{M}_{1,\infty}}$, with*

$$\|A\|_{\mathscr{M}_{1,\infty}} := \sup_{n\geq 1} \frac{1}{\ln(n+1)} \sum_{j=1}^{d} \mu_j(A).$$

Exercise 3.3.5. *Show that for any $A \in \mathscr{J}_1$ one has $\mathrm{Tr}_\omega(A) = 0$.*

As a consequence of the statement contained in the previous exercise, the tracial weight Tr_ω is non-normal, see Definition 2.6.9. Indeed, any approximation of a compact operator by finite rank operators would lead to a trivial trace Tr_ω. It only remains to show that the Tr_ω is a trace on $\mathscr{M}_{1,\infty}$.

Lemma 3.3.6. *For any $A \in \mathscr{M}_{1,\infty}$ and $B \in \mathscr{B}(\mathcal{H})$ one has $\mathrm{Tr}_\omega(AB) = \mathrm{Tr}_\omega(BA)$.*

Proof. Recall that every element of $\mathscr{B}(\mathcal{H})$ can be written has a linear combination of four unitary operators, see for example [Mur, Rem. 2.2.2]. Thus, by linearity it is sufficient to show that $\mathrm{Tr}_\omega(AU) = \mathrm{Tr}_\omega(UA)$ for any unitary $U \in \mathscr{B}(\mathcal{H})$. In addition, since A itself is a linear combination of positive operators, if is sufficient to show the previous equality for positive A. Now, such an equality follows directly from the observation that $\mu_j(AU) = \mu_j(UA) = \mu_j(A)$ for any $j \in \mathbb{N}$. □

Let us finally observe that the trace Tr_ω is a *symmetric* functional in the following sense: If $A, B \in \big(\mathscr{M}_{1,\infty}\big)_+$ satisfy $\mu_n(A) = \mu_n(B)$ for any $n \in \mathbb{N}$, then $\mathrm{Tr}_\omega(A) = \mathrm{Tr}_\omega(B)$.

Remark 3.3.7. *The construction above is based on an invariant states ω and on the use of the function $n \mapsto \ln(n+1)$. It it natural to wonder how much freedom one has for these choices, and how many different Dixmier traces exist ? Deep investigations in that direction have recently been performed and lot's of material has been gathered in [LSZ]. In the next section we present part of this material.*

3.4 Generalizations of the Dixmier trace

In this section we recast the construction of the Dixmier trace in a more general framework, as presented in [SU, SUZ1].

3.4.1 Extended limits

The first step consists in using the more developed theory of extended limits on L^∞ instead of states on ℓ_∞. More precisely, we shall consider $L^\infty(\mathbb{R})$ and $L^\infty(\mathbb{R}_+)$ as the set of essentially bounded Lebesgue measurable functions on $\mathbb{R}$ and $\mathbb{R}_+$ endowed with the norm $\|f\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}}|f(x)|$ or $\|f\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}_+}|f(x)|$ respectively. One aim for considering more general extended limits is to analysis the dependence on ω of the r.h.s. of (3.6).

In analogy to the operations acting on ℓ_∞ we start by introducing the translation operators, namely for any $y \in \mathbb{R}$ we define the operator $T_y : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ by the relation

$$[T_y f](x) := f(x+y), \quad f \in L^\infty(\mathbb{R}).$$

We can now set:

Definition 3.4.1. *A linear functional ω on $L^\infty(\mathbb{R})$ is called* a translation invariant extended limit on $L^\infty(\mathbb{R})$ *if the following conditions are satisfied:*

(i) ω is positive, i.e. $\omega(f) \geq 0$ *whenever* $f \in L_\infty$ *satisfies* $f \geq 0$*,*

(ii) $\omega(\mathbf{1}) = 1$ where $\mathbf{1}$ is the constant function equal to 1 in $L^\infty(\mathbb{R})$,

(iii) $\omega(\chi_{(-\infty,0)}) = 0$ where $\chi_{(-\infty,0)}$ corresponds to the characteristic function on $\mathbb{R}_-$,

(iv) $\omega(T_y f) = \omega(f)$ for every $y \in \mathbb{R}$ and $f \in L^\infty(\mathbb{R})$.

Let us note that a more appropriate name would be an *extended limit at* $+\infty$ since the behavior near $-\infty$ does not really matter.

Exercise 3.4.2. *Show that if* $\lim_{x\to\infty} f(x)$ *exists, then one has* $\omega(f) = \lim_{x\to\infty} f(x)$*, which justifies the name* extended limit*. For that purpose, one can start by showing that if $f \in L^\infty(\mathbb{R})$ has support on $\mathbb{R}_-$, then $\omega(f) = 0$.*

The following functional has been introduced and studied in [SUZ1, Sec. 3]. For any real-valued $f \in L^\infty(\mathbb{R})$ we set

$$p_T(f) := \lim_{x\to\infty} \sup_{h\geq 0} \frac{1}{x}\int_0^x f(y+h)\,\mathrm{d}y \ . \tag{3.12}$$

Note that the index T refers to translation. The main utility of this functional is contained in the following statements, whose proofs are given in [SUZ1, Thms. 13 & 14].

Theorem 3.4.3. *For any uniformly continuous and real-valued function $f \in L^\infty(\mathbb{R})$ the following equality holds:*

$$[-p_T(-f), p_T(f)] = \big\{\omega(f) \mid \omega \text{ is a translation invariant extended limit on } L^\infty(\mathbb{R})\big\}.$$

Note that the assumption about uniform continuity is necessary. As a consequence, one infers a continuous analogue of the classical result on extended limits of [Lor].

Theorem 3.4.4. *Let f be a uniformly continuous and real-valued function $f \in L^\infty(\mathbb{R})$ and let $c \in \mathbb{R}$. The equality $\omega(f) = c$ holds for every translation invariant extended limits on $L^\infty(\mathbb{R})$ if and only if*

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(y+h)\,\mathrm{d}y = c$$

uniformly in $h \geq 0$.

Extension 3.4.5. *Study the previous two theorems and their proof.*

Let us now switch from extended limits on $L^\infty(\mathbb{R})$ to extended limits on $L^\infty(\mathbb{R}_+)$. Again, by analogy with the operation acting on ℓ_∞ we can introduce the dilation operator by $\beta > 0$ by $\sigma_{1/\beta} : L^\infty(\mathbb{R}_+) \to L^\infty(\mathbb{R}_+)$ defined by

$$[\sigma_{1/\beta} f](x) := f(\beta x), \quad f \in L^\infty(\mathbb{R}_+).$$

In this framework, the notion of dilation invariant extended limit is provided by:

Definition 3.4.6. *A linear functional ω on $L^\infty(\mathbb{R}_+)$ is called* a dilation invariant extended limit on $L^\infty(\mathbb{R}_+)$ *if the following conditions are satisfied:*

(i) ω is positive,

(ii) $\omega(\mathbf{1}) = 1$ where $\mathbf{1}$ is the constant function equal to 1 in $L^\infty(\mathbb{R}_+)$,

(iii) $\omega(\chi_{(0,1)}) = 0$ where $\chi_{(0,1)}$ corresponds to the characteristic function on $(0,1)$,

(iv) $\omega(\sigma_{1/\beta} f) = \omega(f)$ for every $\beta > 0$ and $f \in L^\infty(\mathbb{R}_+)$.

Obviously, Definitions 3.4.1 and 3.4.6 have been chosen such that there is a one-to-one correspondence between them. Indeed if ω is a translation invariant extended limit on $L^\infty(\mathbb{R})$, then the linear functional $\exp^* \omega$ defined on any $f \in L^\infty(\mathbb{R}_+)$ by $[\exp^* \omega](f) := \omega(f \circ \exp)$ is a dilation invariant extended limit on $L^\infty(\mathbb{R}_+)$. The converse statement also holds, by using a logarithmic function.

Exercise 3.4.7. *Fix the details of the previous observation.*

By analogy to (3.12) it is now natural to introduce the functional on any real-valued $f \in L^\infty(\mathbb{R}_+)$ by

$$p_D(f) := \lim_{x\to\infty} \sup_{\beta \geq 1} \frac{1}{\ln(x)} \int_1^x f(\beta y) \frac{\mathrm{d}y}{y} \ . \tag{3.13}$$

From the previous correspondence and from Theorems 3.4.3 and 3.4.4 one directly deduces that:

Theorem 3.4.8. *For any real $f \in L^\infty(\mathbb{R}_+)$ such that $f \circ \exp$ is uniformly continuous on $\mathbb{R}$, the following equality holds:*

$$[-p_D(-f), p_D(f)] = \big\{\omega(f) \mid \omega \text{ is a dilation invariant extended limit on } L^\infty(\mathbb{R}_+)\big\}.$$

Theorem 3.4.9. *Let $f \in L^\infty(\mathbb{R})$ be real and such that $f \circ \exp$ is a uniformly continuous function on $\mathbb{R}$, and let $c \in \mathbb{R}$. The equality $\omega(f) = c$ holds for every dilation invariant extended limits on $L^\infty(\mathbb{R}_+)$ if and only if*

$$\lim_{x\to\infty} \frac{1}{\ln(x)} \int_1^x f(\beta y)\frac{\mathrm{d}y}{y} = c$$

uniformly in $\beta \geq 1$.

3.4.2 Additional spaces on $\mathbb{R}_+$

In this subsection we mention the analogue of the sequence spaces introduced in Section 3.2 but in the continuous setting. As a first step and in order to take benefit of $\mathbb{R}_+$ instead of $\mathbb{N}$ let us provide an extension of the function μ giving the singular values of any $A \in \mathscr{K}(\mathcal{H})$. More precisely, for any $A \in \mathscr{K}(\mathcal{H})$ let us set

$$\mu(\cdot, A) := \sum_{j=1}^{\infty} \mu_j(A)\chi_{(j-1,j]}(\cdot) \tag{3.14}$$

Clearly, this function is non-increasing and satisfies the equality $\mu(n, A) = \mu_n(A)$ for any $n \in \mathbb{N}$. It is natural to call $\mu(\cdot, A)$ the *singular values function* of A.

Remark 3.4.10. *A slightly different but more common definition for this function could be given by*

$$\mu(t, A) := \inf\big\{s \geq 0 \mid \mathrm{Tr}\big(\chi_{(s,\infty)}(|A|)\big) \leq t\big\} \tag{3.15}$$

where $\chi_{(s,\infty)}(|A|)\big)$ denotes the spectral projection associated with $|A|$ on the interval (s, ∞). Clearly, $\mathrm{Tr}\big(\chi_{(s,\infty)}(|A|)\big)$ gives the number of eigenvalues of $|A|$ inside the interval (s, ∞) multiplicity counted. Thus, for a given $t > 0$ the r.h.s. of (3.14) *provides the minimal value s such that $|A|$ has t eigenvalues in the interval (s, ∞). With the notation of* (3.14) *this function is equal to $\sum_{j=1}^{\infty} \mu_j(A)\chi_{[j-1,j)}(\cdot)$, and thus $\mu(n, A)$ would not be equal to $\mu_n(A)$ but to $\mu_{n+1}(A)$. By changing our convention on the index of the singular values (and starting with $\mu_0(A)$ instead of $\mu_1(A)$), one could have used* (3.15)*. Note that the interest in* (3.15) *is that it extends quite straightforwardly to the more general context of semi-finite von Neumann algebra endowed with a semi-finite normal trace, see [LSZ] for this general framework.*

Let us now denote by $\Psi^{\mathrm{inc}}_{\mathrm{con}}(\mathbb{R}_+)$ the set of increasing and concave functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{x\to 0} \psi(x) = 0$ and $\lim_{x\to\infty} \psi(x) = \infty$. In the present context and for any $\psi \in \Psi^{\mathrm{inc}}_{\mathrm{con}}(\mathbb{R}_+)$ it is natural to define the Lorentz ideal $\mathscr{M}_\psi$ by

$$\mathscr{M}_\psi = \Big\{A \in \mathscr{K}(\mathcal{H}) \mid \|A\|_\psi := \sup_{x>0} \frac{1}{\psi(x)} \int_0^x \mu(y, A)\,\mathrm{d}y < \infty\Big\}. \tag{3.16}$$

Also, when $\psi(x) = \ln(1 + x)$ the Lorentz ideal will be denoted by $\mathscr{M}_{1,\infty}$. This ideal is sometimes called *the Dixmier ideal.* The spaces $\mathscr{L}_{p,\infty}$ are then defined for any $p \geq 1$ by

$$\mathscr{L}_{p,\infty} = \big\{A \in \mathscr{K}(\mathcal{H}) \mid \sup_{x>0} x^{1/p}\mu(x, A) < \infty\big\}. \tag{3.17}$$

3.4.3 Dixmier traces

In this subsection we generalize the construction of Dixmier by considering dilation invariant extended limits on $\mathbb{R}_+$. Recall that the notion of weight has been introduced in Definition 2.6.7 and corresponds to a positive homogeneous and additive functional.

Definition 3.4.11. *Let ω be a dilation invariant extended limit on $L^\infty(\mathbb{R}_+)$ and let $\psi \in \Psi_{\rm con}^{\rm inc}(\mathbb{R}_+)$. If the functional $\mathrm{Tr}_\omega : (\mathscr{M}_\psi)_+ \to [0,\infty)$ defined on $A \in (\mathscr{M}_\psi)_+$ by*

$$\mathrm{Tr}_\omega(A) := \omega\Big(x \mapsto \frac{1}{\psi(x)} \int_0^x \mu(y, A)\,\mathrm{d}y\Big) \tag{3.18}$$

is a weight on $\mathscr{M}_\psi$, then its extension by linearity on $\mathscr{M}_\psi$ is called a Dixmier trace on *$\mathscr{M}_\psi$.*

Based on a rather deep analysis, the following result has been proved in [DPSS, Thm. 3.4] and in [LSZ, Thm.6.3.3]. Note that the result is in fact proved in a slightly more general context, namely without referring to compact operators and to the specific functions $\mu(\cdot, A)$. In addition, more precise information on the functional Tr_ω are provided in [DPSS].

Theorem 3.4.12. *The Lorentz ideal $\mathscr{M}_\psi$ admits non-trivial Dixmier traces if and only if the function $\psi \in \Psi_{\rm con}^{\rm inc}(\mathbb{R}_+)$ satisfies the additional condition*

$$\liminf_{x\to\infty} \frac{\psi(2x)}{\psi(x)} = 1. \tag{3.19}$$

Before going on, let us compare this result with the result obtained in the previous section. Here we consider arbitrary $\psi \in \Psi_{\rm con}^{\rm inc}(\mathbb{R}_+)$ while in Section 3.3 only the special case $\psi(x) = \ln(x+1)$ was considered. In addition, the properties

$$\lim_{n\to\infty} \frac{\ln(2n+1)}{\ln(n+1)} = 1$$
$$\frac{1}{\ln(2n+1)} - \frac{1}{\ln(2n)} = o\Big(\frac{1}{\ln(2n+1)}\Big)$$

have been explicitly used in the previous proof. In the result mentioned above, only the condition (3.19) is necessary. In addition, since the above result corresponds to a necessary and sufficient condition it can be considered as a rather deep extension of the construction of Dixmier.

Our next aim is to characterize the dilation invariant extended limits which generate a Dixmier trace on $\mathscr{M}_\psi$. For that purpose, the following definition is useful.

Definition 3.4.13. *For any $\psi \in \Psi_{\rm con}^{\rm inc}(\mathbb{R}_+)$, a dilation invariant extended limit ω on $L^\infty(\mathbb{R}_+)$ is* ψ-compatible *or* compatible with ψ *if*

$$\omega\Big(x \mapsto \frac{\psi(2x)}{\psi(x)}\Big) = 1.$$

With this definition at hand, the following result has been proved in [KSS, Thm. 10] or in [SU, Thm. 2.15].

Theorem 3.4.14. *Let* $\psi \in \Psi_{\mathrm{con}}^{\mathrm{inc}}(\mathbb{R}_+)$ *satisfying condition* (3.19). *Let* ω *be a dilation invariant extended limit on* $L^\infty(\mathbb{R}_+)$ *which is compatible with* ψ. *Then the functional* Tr_ω *defined by* (3.18) *on* $(\mathscr{M}_\psi)_+$ *defines a non-normal Dixmier trace.*

In fact, a stronger statement has been proved in these references. First of all, if the functional defined by (3.18) defines a Dixmier trace, then the corresponding state ω is ψ-compatible. In addition, to any normalized fully symmetric functional φ on $\mathscr{M}_\psi$ one can associate a dilation invariant extended limit on $L^\infty(\mathbb{R}_+)$ such that $\mathrm{Tr}_\omega = \varphi$. Since the notion of fully symmetric has not been introduced here (but corresponds to the property appearing in Theorem 2.3.11.(b) in the restricted setting of Section 2) we shall not go further in this direction.

It is now time to show that the continuous approach considered in this section coincides with the discrete approach of Section 3.3.

Exercise 3.4.15. *Show that if* $\psi(x) = \ln(x+1)$, *then Theorem 3.3.1 and the results presented in this section are equivalent.*

Up to now, one question has not been discussed: how many values can one generate by $\mathrm{Tr}_\omega(A)$ for different dilation invariant extended limits ω ? In order to answer this question, let us first introduce the following definition:

Definition 3.4.16. *Let* $\psi \in \Psi_{\mathrm{con}}^{\mathrm{inc}}(\mathbb{R}_+)$ *satisfying condition* (3.19). *An operator* $A \in \mathscr{M}_\psi$ *is called* Dixmier measurable *if all values of the Dixmier traces* $\mathrm{Tr}_\omega(A)$ *coincide.*

Let us also recall that for any $A \in \mathscr{M}_\psi$ one has the unique decomposition $A = A_1 - A_2 + iA_3 - iA_4$ with each $A_j \in (\mathscr{M}_\psi)_+$. It then follows that

$$\begin{aligned}\mathrm{Tr}_\omega(A) &= \mathrm{Tr}_\omega(A_1) - \mathrm{Tr}_\omega(A_2) + i\mathrm{Tr}_\omega(A_3) - i\mathrm{Tr}_\omega(A_4)\\ &= \omega\Big(x \mapsto \frac{1}{\psi(x)}\int_0^x \big(\mu(y,A_1) - \mu(y,A_2) + i\mu(y,A_3) - i\mu(y,A_4)\big)\,\mathrm{d}y\Big)\\ &= \omega\Big(x \mapsto \frac{1}{\psi(x)}\int_0^x \tilde\mu(y,A)\,\mathrm{d}y\Big)\end{aligned}$$

with $\tilde\mu(y,A) := \mu(y,A_1) - \mu(y,A_2) + i\mu(y,A_3) - i\mu(y,A_4)$. Since the function

$$x \mapsto \frac{1}{\psi(x)}\int_0^x \tilde\mu(y,A)\,\mathrm{d}y$$

is absolutely continuous, we can then use the criterion introduced in Theorem 3.4.9 and infer (see also [SUZ2, Cor. 3.2]) :

Theorem 3.4.17. *Let* $\psi \in \Psi_{\mathrm{con}}^{\mathrm{inc}}(\mathbb{R}_+)$ *satisfying condition* (3.19). *An element* $A \in \mathscr{M}_\psi$ *is Dixmier measurable if and only if the limit*

$$\lim_{x\to\infty}\frac{1}{\ln(x)}\int_1^x \Big(\frac{1}{\psi(\beta y)}\int_0^{\beta y}\tilde\mu(z,A)\,\mathrm{d}z\Big)\frac{\mathrm{d}y}{y}$$

exists uniformly in $\beta \geq 1$. *If so,* $\mathrm{Tr}_\omega(A)$ *is equal to this limit for all Dixmier traces.*

Let us finally mention that for positive operators, the above condition can be simplified, but the condition on ψ is slightly more restrictive.

Theorem 3.4.18. *Let $\psi \in \Psi_{\rm con}^{\rm inc}(\mathbb{R}_+)$ satisfying the condition $\lim_{x\to\infty} \frac{\psi(2x)}{\psi(x)} = 1$. An element $A \in (\mathscr{M}_\psi)_+$ is Dixmier measurable if and only if the limit*

$$\lim_{x\to\infty} \frac{1}{\psi(x)} \int_0^x \mu(y, A)\,\mathrm{d}y$$

exists. If so, $\mathrm{Tr}_\omega(A)$ is equal to this limit for all Dixmier traces.

As a final remark, let us recall that this theory can be applied to a large class of von Neumann algebra instead of $\mathscr{B}(\mathcal{H})$. However, one has to be cautious with the hypotheses in all the statements since counterexamples have been constructed for checking the optimality of several results. Some of them are recalled in the reference [SU] which has been the main source of inspiration for this section.