Nagoya University, G30 program Spring 2017

Mathematics Tutorial IIa (calculus) Instructor : Serge Richard

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Homework 7

Exercise 1 *Let us consider* $f:\mathbb{R}^3\to\mathbb{R}$ *defined by* $f(x,y,z)=\mathrm{e}^{xy}\cos(z)$ *for any* $(x,y,z)\in\mathbb{R}^3$*. Assume also that* $x=tu$*,* $y=\sin(tu)$ *and* $z=u^2$ *for some* $t,u\in\mathbb{R}$*. By setting*

$$F(t,u):=f\big(tu,\sin(tu),u^2\big)$$

Compute the derivative $\partial_2 F\equiv\partial_u F$ *by three different methods: once by a direct computation, once as one component of the derivative of the composition of two functions (chain rule), and once with the formula often seen in the literature*

$$\frac{\partial F}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial u}.$$

Can you explain where this formula comes from ?

Exercise 2 (Spherical coordinates) *Consider the map* $\Phi:[0,\infty)\times[0,2\pi)\times[0,\pi)\to\mathbb{R}^3$ *with*

$$\Phi(r,\theta,\varphi):=\big(r\cos(\theta)\sin(\varphi),r\sin(\theta)\sin(\varphi),r\cos(\varphi)\big).$$

Compute the Jacobian matrix corresponding to this function.

Recall that *a vector field* is a function f defined on an open set $\Omega\subset\mathbb{R}^n$ and taking values in $\mathbb{R}^n$. For such a vector field f, if there exists a differentiable function $\phi:\Omega\to\mathbb{R}$ such that $f=\nabla\phi$ we say that ϕ is a *potential function for* f.

Exercise 3 *For the following functions, does it exist a potential function ?*

(i) $f(x,y)=(y,x)$,

(ii) $f(x,y)=(3x^2y+2x+y^3,x^3+3xy^2-2y)$,

(iii) $f(x,y)=\big(\cos(x),\sin(y)\big)$,

(iv) $f(x,y,z)=\big(x^2-yz,y^2-zx,z^2-xy\big)$.

Exercise 4 *a) Let* $\Omega\subset\mathbb{R}^2$ *be open and let* $F:\Omega\to\mathbb{R}^2$ *be of class* C^1 *on* Ω*. Let us also set* $F=(f_1,f_2)$*. Show that if*

$$\partial_x f_2\neq\partial_y f_1$$

then F *does not admit a potential function of class* C^2*.*

b) What would be a similar statement for a function $F:\Omega\to\mathbb{R}^3$ *if* Ω *is an open subset of* $\mathbb{R}^3$*.*

c) What about the n*-dimensional case, and how many conditions have to be satisfied ?*

Exercise 5 *Consider the vector field* $F:\mathbb{R}^2\setminus\{(0,0)\}\to\mathbb{R}^2$ *defined for* $(x,y)\neq(0,0)$ *by*

$$F(x,y)=\Big(\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2}\Big)$$

(i) Represent graphically this vector field (you can use polar coordinates),

(ii) Can you find a potential function for this vector field, and if so exhibit it.