Nagoya University, G30 program Spring 2017

Mathematics Tutorial IIa (calculus) Instructor : Serge Richard

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Homework 4

Let us recall the polar coordinates $(r, \theta) \in [0, \infty) \times [0, 2\pi)$ which allow one to give an alternative description of all the points of $\mathbb{R}^2$. The relations between the usual coordinates $(x, y) \in \mathbb{R} \times \mathbb{R}$ are

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta).$$

Exercise 1 *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) := \begin{cases} \frac{x^3}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

1. *By using the polar coordinates, study the continuity of f at $(0,0)$,*
2. *Show that f is differentiable with continuous partial derivatives on $\mathbb{R}^2 \setminus \{(0,0)\}$,*
3. *Show that f admits in $(0,0)$ derivatives in all directions,*
4. *Show that f is not differentiable at $(0,0)$.*

Exercise 2 *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) := \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

and compute its second derivatives $\partial_j \partial_k f$ for any $j, k \in \{x, y\}$. Are these functions continuous ?

Exercise 3 (Geometrical interpretation of the gradient) *a) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^2 + 4y^2$.*

(i) *Compute the gradient of f at any point (x,y),*

(ii) *For $k > 0$, describe the k-level set L_k, and for this level set express y as a function of x and k,*

(iii) *For any $(x,y) \in \mathbb{R}^2$ such that $f(x,y) = k$, show that the gradient of f at (x,y) is orthogonal to the curve described by L_k.*

b) More generally, let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^1 and let $X \in \mathbb{R}^n$ such that $[\nabla f](X) \neq \mathbf{0}$. Let $k \in \mathbb{R}$ be given by $k := f(X)$ and consider the k-level set L_k. This k-level set can be considered (at least locally) as a surface of dimension $n-1$ in $\mathbb{R}^n$. Show that $[\nabla f](X)$ is perpendicular to the surface L_k. For that purpose, we can consider any parametric curve $\varphi : (-1,1) \to L_k$ with $\varphi(0) = X$ and show that $[\nabla f](X)$ is perpendicular to it at the point X.

Exercise 4 (The chain rule) *Let Ω be an open set in $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}$ be differentiable. Let (a,b) be an open interval in $\mathbb{R}$, and consider $\varphi : (a,b) \to \Omega$ be a parametric curve which is differentiable. Show that the following equality holds:*

$$f\big(\varphi(t)\big)' := \frac{\mathrm{d}f\big(\varphi(t)\big)}{\mathrm{d}t} = [\nabla f](\varphi(t)) \cdot \varphi'(t).$$