The report for the class "Hilbert space methods for quantum mechanics"

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This report is about the decomposition of a Hilbert space with respect to a self-adjoint operator (Exercise 4.3.4) and Stone's theorem. Let us introduce some notations.

For $z \in \mathbb{C}$, we denote by $\mathrm{Re}z(\mathrm{Im}z)$ the real(imaginary) part of z, by $\mathcal{A}_B$ the set of all Borel sets on $\mathbb{R}$, the Lebesgue measure by $|\cdot|$, and the set of all bounded continuous maps on $\mathbb{R}$ by $C_b(\mathbb{R})$. Moreover, for a Hilbert space $\mathcal{H}$, a subspace M of $\mathcal{H}$ and an operator A, we denote the identity map by I, the orthogonal complement of M in $\mathcal{H}$ by $M^{\perp_{\mathcal{H}}}$, the domain(range) of A by $D(A)(\mathrm{Ran}(A))$, the spectrum(resolvent set) of A by $\sigma(A)(\rho(A))$. In particular, $M^{\perp_{\mathcal{H}}} = \{f \in \mathcal{H} \mid \forall g \in M, \langle f, g\rangle = 0\}$, $\mathrm{Ran}(A) = AD(A)$. Futhermore, let the inner product be linear in the second argument.

We begin with the result that a Hilbert space $\mathcal{H}$ can be decomposed with respect to a self-adjoint operator A. Let $\{E_\lambda\}$ be a spectral family. We define E as

$$E((a,b]) := E_b - E_a \quad (a < b,\ a, b \in \mathbb{R})$$

and extend this definition to all set $V \in \mathcal{A}_B$. For all $f \in \mathcal{H}$, set

$$F_f(\lambda) := \langle f, E_\lambda f\rangle = \|E_\lambda f\|^2 \quad (\lambda \in \mathbb{R}).$$

This function is non-decreasing, right continuous and satisfies

$$F_f(\lambda) \to 0 \text{ as } \lambda \to -\infty \quad \text{and} \quad F_f(\lambda) \to \|f\|^2 < \infty \text{ as } \lambda \to \infty.$$

Hence, we can make a measure m_f on $\mathbb{R}$ such that

$$m_f((a,b]) = \langle f, E((a,b])f\rangle \quad (a < b,\ a, b \in \mathbb{R}).$$

Let φ be a continuous function on $\mathbb{R}$. For any $M > 0$ and partition of $(-M, M]$:

$$\Pi = \{s_0, \cdots, s_N; u_1, \cdots, u_N\} \text{ with } -M = s_0 < u_1 < s_1 < \cdots < u_N < s_N = M,$$

we define the Riemann sum Σ_Π^M and the mesh $d(\Pi)$ of Π as

$$\Sigma_\Pi^M := \sum_{k=1}^{N} \varphi(u_k) E((s_{k-1}, s_k]), \qquad d(\Pi) := \max_{1 \le k \le N}(s_k - s_{k-1})$$

and then, we define $(\int_{-\infty}^{\infty} \varphi(\lambda)E(d\lambda), D_\varphi)$ as

$$D_\varphi := \left\{ f \in \mathcal{H} \;\middle|\; \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 m_f(d\lambda) < \infty \right\}, \quad \int_{-\infty}^{\infty} \varphi(\lambda)E(d\lambda) := \text{s-}\lim_{M\to\infty} \text{s-}\lim_{d(\Pi)\to 0} \Sigma_\Pi^M.$$

By the spectrum theorem, we can associate a unique spectral family $\{E_\lambda\}$ (called the spectral family of A) such that $(A, D(A)) = (\int_{-\infty}^{\infty} \lambda E(d\lambda), D_{\mathrm{id}})$.

We take $\{E_\lambda\}$ as the spectral family of A below. In connection with this theorem, we define bounded operator $\varphi(A)$ as $\int_{-\infty}^{\infty} \varphi(\lambda)E(d\lambda)$ for $\varphi \in C_b(\mathbb{R})$. We obtain,

$$\left\| \int_{-\infty}^{\infty} \varphi(\lambda)E(d\lambda)f \right\|^2 = \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 m_f(d\lambda) \qquad (f \in \mathcal{H}), \tag{1}$$

$$\int_{-\infty}^{\infty} \varphi(\lambda)E(d\lambda) \cdot \int_{-\infty}^{\infty} \psi(\lambda)E(d\lambda) = \int_{-\infty}^{\infty} \varphi(\lambda)\psi(\lambda)E(d\lambda) \qquad (\psi \in C_b(\mathbb{R})). \tag{2}$$

We note that the eigenvalues of the self-adjoint operator A are countable.

Definition 1. Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and E, m_f be above.

$$\mathcal{H}_p(A) := \bigoplus_{\lambda} \mathrm{Ran}(E(\{\lambda\})) = \{f \in \mathcal{H} \mid \exists \mathcal{V} : \text{countable},\ E(\mathcal{V})f = f\} \tag{3}$$

$$\begin{aligned} \mathcal{H}_{sc}(A) &:= \{f \in \mathcal{H} \mid m_f \text{ is singular continuous}\} \\ &= \{f \in \mathcal{H} \mid \exists \mathcal{V} \in \mathcal{A}_B, |\mathcal{V}| = 0, E(\mathcal{V})f = f \text{ and } \forall \lambda \in \mathbb{R}, E(\{\lambda\})f = 0\} \end{aligned} \tag{4}$$

$$\begin{aligned} \mathcal{H}_{ac}(A) &:= \{f \in \mathcal{H} \mid m_f \text{ is absolutely continuous}\} \\ &= \{f \in \mathcal{H} \mid \forall \mathcal{V} \in \mathcal{A}_B, |\mathcal{V}| = 0 \Longrightarrow E(\mathcal{V})f = 0\} \end{aligned} \tag{5}$$

$$\mathcal{H}_c(A) := \{f \in \mathcal{H} \mid m_f \text{ is continuous}\} = \{f \in \mathcal{H} \mid \forall \lambda \in \mathbb{R}, E(\{\lambda\})f = 0\} \tag{6}$$

Note that the right-hand side of (3)-(6) is closed. Let us verify for example that the right-hand side of (4) is a subspace, and in particular closed. Assume

$$\forall j \in \mathbb{N}, \quad f_j \in \mathcal{H}, \quad \exists \mathcal{V}_j \in \mathcal{A}_B, |\mathcal{V}_j| = 0, E(\mathcal{V}_j)f_j = f_j, \quad \forall \lambda \in \mathbb{R}, E(\{\lambda\})f_j = 0$$

and $f_j \to f$ as $j \to \infty$. Set $\mathcal{V} = \bigcup_j \mathcal{V}_j$. $E(\mathcal{V})f_j = f_j$ holds. Since $E(\mathcal{V}), E(\{\lambda\})$ is an orthogonal projection, $E(\mathcal{V})f = f, E(\{\lambda\})f = 0$ for any $\lambda \in \mathbb{R}$.

Firstly, we prove (5). Let $f \in \mathcal{H}$. If m_f is absolutely continuous, for any $\mathcal{V} \in \mathcal{A}_B$ satisfying $|\mathcal{V}| = 0$, $\|E(\mathcal{V})f\|^2 = \langle f, E(\mathcal{V})f\rangle = m_f(\mathcal{V}) = 0$, and so $E(\mathcal{V})f = 0$. The converse holds similarly: by $m_f(\mathcal{V}) = \|E(\mathcal{V})f\|^2$. (4), (6) are also proved similarly. We shall prove (3) i.e. $\bar{D} = \mathcal{G}$ where $D := \{\sum_{j=1}^n c_j f_j \mid c_j \in \mathbb{C},\ f_j \in \mathrm{Ran}(E(\{\lambda_j\})),\ \lambda_j :$ eigenvalue$\}$, $\mathcal{G} :=$ (the right-hand side of (3)). For all eigenvalue λ and $f \in E(\{\lambda\})$, $f \in \mathcal{G}$ (take $\mathcal{V} = \{\lambda\}$), and thus $D \subset \mathcal{G}$. Moreover, since $\mathcal{G}$ is closed, $\bar{D} \subset \mathcal{G}$. Conversely, let $f \in \mathcal{G}$. Take $\mathcal{V} = \{\lambda_j\}_{j=1}^{\infty}$ such that $E(\mathcal{V})f = f$. Since $E(\{\lambda_j\})f \in \mathrm{Ran}(E(\{\lambda_j\}))$, $f = \lim_{n\to\infty} E(\{\lambda_1, \cdots, \lambda_n\})f = \lim \big(E(\{\lambda_1\})f + \cdots + E(\{\lambda_n\})f\big) \in \bar{D}$. Thus, $\bar{D} = \mathcal{G}$.

Note that for any $f \in \mathcal{H}_c(A)$ (particularly, for any $f \in \mathcal{H}_{sc}(A)$) and any countable set $\mathcal{V} \subset \mathbb{R}$, we have $E(\mathcal{V})f = 0$.

Theorem 1. Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$.

a) $\mathcal{H}$ can be decomposed as follows:

$$\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_{sc}(A) \oplus \mathcal{H}_{ac}(A). \tag{7}$$

b) Denote the restriction of the operator A to each of these subspaces by A_p, A_{sc}, A_{ac} (for example $D(A_p) = D(A) \cap \mathcal{H}_p,\ A_p f = Af$ for $f \in D(A_p)$). These operators are self-adjoint as operators in $\mathcal{H}_p(A), \mathcal{H}_{sc}(A), \mathcal{H}_{ac}(A)$ respectively, and we have

$$A = A_p \oplus A_{sc} \oplus A_{ac}, \quad \varphi(A) = \varphi(A_p) \oplus \varphi(A_{sc}) \oplus \varphi(A_{ac}) \ \ (\forall \varphi \in C_b(\mathbb{R})). \tag{8}$$

c) $\sigma(A) = \sigma(A_p) \cup \sigma(A_{sc}) \cup \sigma(A_{ac})$.

Proof. a) It suffices to prove $\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_c(A)$ and $\mathcal{H}_c(A) = \mathcal{H}_{sc}(A) \oplus \mathcal{H}_{ac}(A)$, i.e. (i)$\mathcal{H}_c(A) = \mathcal{H}_p(A)^{\perp \mathcal{H}}$ and (ii) $\mathcal{H}_{ac}(A) = \mathcal{H}_{sc}(A)^{\perp \mathcal{H}_c(A)}$. We prove (i). For any $f \in \mathcal{H}_c(A)$ and $g \in \mathcal{H}_p(A)$, taking a countable set $\mathcal{V} \subset \mathbb{R}$ such that $E(\mathcal{V})g = g$, we get $\langle g, f \rangle = \langle E(\mathcal{V})g, f \rangle = \langle g, E(\mathcal{V})f \rangle = 0$. On the other hand, for any $f \in \mathcal{H}_p(A)^{\perp \mathcal{H}}$, $g \in \mathcal{H}$ and $\lambda \in \mathbb{R}$, by $E(\{\lambda\})g \in \mathrm{Ran}(E(\{\lambda\})) \subset \mathcal{H}_p(A)$, $\langle g, E(\{\lambda\})f \rangle = \langle E(\{\lambda\})g, f \rangle = 0$, and so $E(\{\lambda\})f = 0$. Thus, (i) holds. (ii) can be proved similarly.

b) By commutativity of the spectral family

$$\forall \mathcal{V} \in \mathcal{A}_B,\ E(\mathcal{V})\mathcal{H}_p(A) \subset \mathcal{H}_p(A), E(\mathcal{V})\mathcal{H}_{sc}(A) \subset \mathcal{H}_{sc}(A), E(\mathcal{V})\mathcal{H}_{ac}(A) \subset \mathcal{H}_{ac}(A). \tag{9}$$

Moreover, we can show $\mathrm{Ran}(A_p) \subset \mathcal{H}_p(A)$, $\mathrm{Ran}(A_{sc}) \subset \mathcal{H}_{sc}(A)$, $\mathrm{Ran}(A_{ac}) \subset \mathcal{H}_{ac}(A)$. We prove $\mathrm{Ran}(A_{sc}) \subset \mathcal{H}_{sc}$ for example. For any $f \in D(A_{sc})$, take $\mathcal{V} \in \mathcal{A}_B$ such that $|\mathcal{V}| = 0$ and $E(\mathcal{V})f = f$. Letting Σ_Π^M be the Riemann sum and $\lambda \in \mathbb{R}$, by commutativity of the spectral family, we have $E(\mathcal{V})\Sigma_\Pi^M f = \Sigma_\Pi^M E(\mathcal{V})f = \Sigma_\Pi^M f$ and $E(\{\lambda\})\Sigma_\Pi^M f = \Sigma_\Pi^M E(\{\lambda\})f = 0$. Hence, letting $d(\Pi) \to 0$ and $M \to \infty$, by boundedness of $E(\mathcal{V})$ and $E(\{\lambda\})$, we get $E(\mathcal{V})Af = Af$ and $E(\{\lambda\})Af = 0$. Thus, we got $Af \in \mathcal{H}_{sc}(A)$.

To prove (8), letting $\varphi = \mathrm{id}$ or $\varphi \in C_b(\mathbb{R})$, we will show (b-i) $D(\varphi(A)) = D(\varphi(A_p)) \oplus D(\varphi(A_{sc})) \oplus D(\varphi(A_{ac}))$ and (b-ii) $f = f_p + f_{sc} + f_{ac},\ f_p \in D(\varphi(A_p)), f_{sc} \in D(\varphi(A_{sc})), f_{ac} \in D(\varphi(A_{ac})) \implies Af = A_p f_p + A_{sc} f_{sc} + A_{ac} f_{ac}$. Since the orthogonality of the decomposition of (b-i) and (b-ii) is obvious, it suffices to prove that $\forall f \in D(\varphi(A))$, $\exists f_p \in D(\varphi(A_p)), \exists f_{sc} \in D(\varphi(A_{sc})), \exists f_{ac} \in D(\varphi(A_{ac})),\ f = f_p + f_{sc} + f_{ac}$. Take f_p, f_{sc}, f_{ac} as the projection of f to $\mathcal{H}_p(A), \mathcal{H}_{sc}(A), \mathcal{H}_{ac}(A)$ respectively. If we have $f_p, f_{sc}, f_{ac} \in D(\varphi(A))$, (8) will be completed. By (9) and the orthogonality of (7),

$$\begin{aligned}
\forall \mathcal{V} \in \mathcal{A}_B,\ \ m_f(\mathcal{V}) &= \langle f_p + f_{sc} + f_{ac}, E(\mathcal{V})(f_p + f_{sc} + f_{ac}) \rangle \\
&= \langle f_p, E(\mathcal{V})f_p \rangle + \langle f_{sc}, E(\mathcal{V})f_{sc} \rangle + \langle f_{ac}, E(\mathcal{V})f_{ac} \rangle \\
&= m_{f_p}(\mathcal{V}) + m_{f_{sc}}(\mathcal{V}) + m_{f_{ac}}(\mathcal{V}).
\end{aligned}$$

Therefore, we can conclude $f_p, f_{sc}, f_{ac} \in D(\varphi(A))$ by

$$\begin{aligned}
&\int_{-\infty}^{\infty} |\varphi(\lambda)|^2\, m_{f_p}(d\lambda) + \int_{-\infty}^{\infty} |\varphi(\lambda)|^2\, m_{f_{sc}}(d\lambda) + \int_{-\infty}^{\infty} |\varphi(\lambda)|^2\, m_{f_{ac}}(d\lambda) \\
&\qquad\qquad = \int_{-\infty}^{\infty} |\varphi(\lambda)|^2\, m_f(d\lambda) < \infty.
\end{aligned}$$

We prove the self-adjointness of A_p. A_{sc} and A_{ac} are similar. Since A is self-adjoint, for any $f, g \in \mathcal{H}_p(A)$, $\langle f, A_p g\rangle = \langle f, Ag\rangle = \langle Af, g\rangle = \langle A_p f, g\rangle$, and so A_p is symmetric. Since A is self-adlont again, by (b-i), $\mathcal{H} = (A \pm i)D(A) = (A_p \pm i)D(A_p) + (A_{sc} \pm i)D(A_{sc}) + (A_{ac} \pm i)D(A_{ac})$. Projecting the left-hand side and right-hand side to $\mathcal{H}_p(A)$, we have $\mathcal{H}_p(A) = P_{\mathcal{H}_p(A)}(A_p \pm i)D(A_p)$. Since $P_{\mathcal{H}_p(A)}(A_p \pm i)D(A_p) \subset (A_p \pm i)D(A_p) \subset \mathcal{H}_p(A)$, we get $(A_p \pm i)D(A_p) = \mathcal{H}_p(A)$ and complete the proof.

c) For $f \in \mathcal{H}$, we denote the projection of f to $\mathcal{H}_p(A), \mathcal{H}_{sc}(A), \mathcal{H}_{ac}(A)$ by f_p, f_{sc}, f_{ac} respectively. If we have (c)' $\rho(A) = \rho(A_p) \cap \rho(A_{sc}) \cap \rho(A_{ac})$, we can conclude (by De Morgan's laws) (c). To prove (c)', we shall prove that (c-i) A is invertible if and only if A_p, A_{sc}, A_{ac} all are invertible and (c-ii) $\mathrm{Ran}(A - z) = \mathrm{Ran}(A_p - z) + \mathrm{Ran}(A_{sc} - z) + \mathrm{Ran}(A_{ac} - z)$ for any $z \in \mathbb{C}$. Note that for $f \in \mathcal{H}$, by (c-ii),

$$(A - z)f = (A_p - z)f_p + (A_{sc} - z)f_{sc} + (A_{ac} - z)f_{ac} \quad (z \in \mathbb{C}). \tag{10}$$

Since any element of the left-hand(right-hand) side of (c-ii) can be written as the left-hand(right-hand) side, and so the right-hand(left-hand) side of (10), (c-ii) holds. Finally, we prove (c-i). Assume A is invertible. Since $f \in D(A_p), (A_p-z)f = 0$ implies $(A-z)f = 0$, A_p (and similarly A_{sc} and A_{ac}) is invertible. Assume A_p, A_{sc}, A_{ac} are invertible conversely. By (10), $f \in \mathcal{H}, (A-z)f = 0$ implies, by the orthogonality of (7), $(A_p-z)f_p = 0$, $(A_{sc} - z)f_{sc} = 0$, $(A_{ac} - z)f_{ac} = 0$, and so $f = f_p + f_{sc} + f_{ac} = 0$. □

Next, we prove that there exists a bijective correspondence between self-adjoint operators and evolution groups. We denote $\varphi(A)$ with $\varphi(\lambda) = e^{-i\lambda t}$ by e^{-iAt} for $t \in \mathbb{R}$.

Definition 2. A family $\{U_t\}_{t\in\mathbb{R}}$ of unitary operators on a Hilbert space $\mathcal{H}$ is called an *evolution group* if $t \mapsto U_t$ is strongly continuous and $U_t U_s = U_{t+s}$ for any $t, s \in \mathbb{R}$.

Proposition 1. Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. $\{e^{-iAt}\}$ is an evolution group and satisfies on $D(A)$:

$$\text{s-}\lim_{t\to 0} \frac{i}{t}[e^{-iAt} - I] = A. \tag{11}$$

Proof. By (1) and (2), we have U_t is unitary and $U_t U_s = U_{t+s}$ $(t, s \in \mathbb{R})$. $\{e^{-iAt}\}$ is strongly continuous, since we get by the dominated convergence theorem

$$\|(e^{-iAs} - e^{-iAt})f\|^2 = \int_{-\infty}^{\infty} |e^{-i\lambda s} - e^{-i\lambda t}|^2\, m_f(d\lambda) \to 0 \text{ as } s \to t \quad (\forall f \in \mathcal{H}).$$

Similarly, we obtain (11) by the dominated convergence theorem: we prove for $f \in D(A)$

$$\left\| \left(\frac{i}{t}[e^{-iAt} - I] - A \right) f \right\|^2 = \int_{-\infty}^{\infty} \left| \frac{i}{t}[e^{-i\lambda t} - 1] - \lambda \right|^2 m_f(d\lambda) \tag{12}$$

goes to 0 as $t \to 0$. Since the mean-value theorem shows $|(e^{-ix} - 1)/x| \le 1 \;\; \forall x \in \mathbb{R}$,

$$\left| \frac{i}{t}[e^{-i\lambda t} - 1] - \lambda \right|^2 \le \left(\left| \frac{e^{-i\lambda t} - 1}{\lambda t} \right| |\lambda| + |\lambda| \right)^2 \le 4\lambda^2.$$

Moreover, the map $\lambda \mapsto 4\lambda^2$ is integrable with respect to the measure m_f for $f \in D(A)$, and thus, (12) goes to 0. □

Theorem 2. (Stone's theorem)

Let $\{U_t\}_{t\in\mathbb{R}}$ be an evolution group in a Hilbert space $\mathcal{H}$. Define the *generator* A as

$$D(A) := \{f \in \mathcal{H} \mid \text{s-}\lim_{t\to 0} t^{-1}[U_t - I]f \text{ exists}\}, \quad Af = \text{s-}\lim_{t\to 0} it^{-1}[U_t - I]f$$

Then, we have

a) $D(A)$ is dense in $\mathcal{H}$ and A is self-adjoint.

b) $U_t D(A) \subset D(A)$ for all $t \in \mathbb{R}$ and for any $f \in D(A)$,

$$i\frac{d}{dt}U_t f = AU_t f = U_t Af.$$

c) $U_t = e^{-iAt}$.

Proof. a) To prove the density of $D(A)$, for $z \in \mathbb{C}$ such that $\mathrm{Im} z > 0$, we set

$$R_z := i\int_0^\infty e^{izs}U_s\, ds$$

and prove $\mathrm{Ran}(R_z)$ is dense in $\mathcal{H}$ and $\mathrm{Ran}(R_z) \subset D(A)$.

Firstly, we show $\mathrm{Ran}(R_z)$ is dense in $\mathcal{H}$ i.e. $\forall g \in R_z^{\perp\mathcal{H}},\ g = 0$. Let $f \in \mathcal{H}$. Since

$$0 = \langle g, R_z U_t f\rangle = i\int_0^\infty \langle g, e^{izs}U_{t+s}f\rangle\, ds = ie^{-izt}\int_t^\infty e^{iz\tau}\langle g, U_\tau f\rangle\, d\tau \quad (\forall t \in \mathbb{R}),$$

$\int_t^\infty e^{iz\tau}\langle g, U_\tau f\rangle\, d\tau = 0$. Differentiating this with respect to t, we have $-e^{izt}\langle g, U_t f\rangle = 0$, and letting $t = 0$, we get $\langle g, f\rangle = 0$ and so $g = 0$. Thus, $\mathrm{Ran}(R_z)$ is dense in $\mathcal{H}$. Secondly, we show $\mathrm{Ran}(R_z) \subset D(A)$. However, this is proved because for any $f \in \mathcal{H}$,

$$\begin{aligned} \frac{i}{t}[U_t - I]R_z f &= -\frac{1}{t}\int_0^\infty e^{izs}U_{t+s}f\, ds - \frac{i}{t}R_z f = -\frac{1}{t}e^{-izt}\int_t^\infty e^{iz\tau}U_\tau f\, d\tau - \frac{i}{t}R_z f \\ &= \frac{i}{t}(e^{-izt} - 1)R_z f + \frac{1}{t}e^{-izt}\int_0^t e^{iz\tau}U_\tau f\, d\tau \to zR_z f + f \text{ as } t \to 0. \quad (13) \end{aligned}$$

Therefore, $D(A)$ is dense and by (13), $AR_z f = zR_z f + f$ i.e. $(A - z)R_z f = f$ for any $f \in \mathcal{H}$. In particular (by taking $z = i$), $\mathrm{Ran}(A - i) = \mathcal{H}$. Repeating the argument above for $z \in \mathbb{C}$ such that $\mathrm{Im} z < 0$ and

$$R_z := -i\int_{-\infty}^0 e^{izs}U_s\, ds,$$

we have $\mathrm{Ran}(A + i) = \mathcal{H}$. Moreover, for any $f, g \in D(A)$,

$$\langle Af, g\rangle = \lim_{t\to 0}\langle it^{-1}[U_t - I]f, g\rangle = \lim_{t\to 0}\langle f, -it^{-1}[U_{-t} - I]g\rangle = \langle f, Ag\rangle,$$

and so A is symmetric. Thus, A is self-adjoint.

b) Assume $f \in D(A)$ and $t \in \mathbb{R}$. Since $(\frac{i}{s}[U_s - I])U_t f = U_t(\frac{i}{s}[U_s - I])f \to U_t Af$ strongly, $U_t f \in D(A)$ and $AU_t f = U_t Af$. Moreover, by $(\frac{i}{s}[U_s - I])U_t f = \frac{i}{s}(U_{t+s} - U_t) \to i\frac{d}{dt}U_t f$ strongly, we have $i\frac{d}{dt}U_t f = AU_t f = U_t Af$.

c) Since U_t and e^{-iAt} are bounded, it suffices to prove $f(t) := (U_t - e^{-iAt})f = 0$ for any f in the dense subset $D(A)$. By b) and Proposition.1,

$$\frac{d}{dt}f(t) = \frac{d}{dt}U_t f - \text{s-}\lim_{s\to 0}\frac{1}{s}[e^{-iAs} - I]e^{-iAt}f = -iAU_t f + iAe^{-iAt}f = -iAf(t),$$

and so, $\frac{d}{dt}\|f(t)\|^2 = 2\mathrm{Re}\langle f(t), -iAf(t)\rangle = 0$ (note that $\langle f(t), Af(t)\rangle$ is real number because A is self-adjoint. Since $f(0) = 0$, we can conclude $f(t) = 0$. □

By this proposition and theorem, the existence of the bijective correspondence between self-adjoint operators and evolution groups is proved.

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