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Let A be a unital C^*-algebra.

Set $\mathcal{P}(A) := \{p \in A \mid p \text{ is an orthogonal projection: i.e., } p = p^* = p^2\}$

We consider the following three equivalence relations on $\mathcal{P}(A)$:

- $p \sim_h q$ in $\mathcal{P}(A)$ $\overset{\text{def.}}{\iff}$ $\exists f : [0,1] \longrightarrow \mathcal{P}(A)$: continuous s.t. $f(0) = p$ & $f(1) = q$

- $p \sim_u q$ $\overset{\text{def.}}{\iff}$ $\exists u \in \mathcal{U}(A)$ s.t. $q = u^* p u$ (where $\mathcal{U}(A)$ denotes the set of all unitary elements of A)

- $p \sim_M q$ $\overset{\text{def.}}{\iff}$ $\exists v \in A$ s.t. $p = v^* v$ & $q = v v^*$

The following properties hold.:

$\boxed{\text{Property 0.}}$

- $p \sim_h q$ in $\mathcal{P}(A)$ $\Longrightarrow$ $p \sim_u q$
- $p \sim_u q$ $\Longrightarrow$ $p \sim_M q$

In general, the inverse of the above implications do not hold.
However, they are correct if A is the matrix algebra $M_n(\mathbb{C})$.
That is,

$\boxed{\text{Main fact of this paper}}$

Let p and q be elements of the matrix algebra $M_n(\mathbb{C})$.

① If $p \sim_M q$, then $p \sim_u q$

② If $p \sim_u q$, then $p \sim_h q$ in $\mathcal{P}(M_n(\mathbb{C}))$

We review some facts for the proof of this.

Lemma 1

$\mathcal{U}(M_n(\mathbb{C})) = \mathcal{U}_0(M_n(\mathbb{C}))$

(where $\mathcal{U}_0(A) := \{u \in \mathcal{U}(A) \mid u \sim_h 1 \text{ in } \mathcal{U}(A)\}$)

Lemma. 2

A : unital C^*-algebra and $p, q \in \mathcal{P}(A)$

$p \sim_h q$ in $\mathcal{P}(A) \iff {}^{\exists}u \in \mathcal{U}_0(A)$ s.t. $q = upu^*$

We already proved the above two facts during on Wednesday my lecture.

Now, we show the main statement. :

<u>proof of ①</u>

Assume that there exists $v \in M_n(\mathbb{C})$ such that $p = v^*v$ and $q = vv^*$.

Since p and q are diagonalizable by unitary matrices and their eigen values are 0 or 1,

${}^{\exists}u_1, u_2 \in \mathcal{U}(M_n(\mathbb{C}))$

s.t. $u_1^* p u_1 = \left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{k} & O \\ \hline O & O \end{array}\right)$ and $u_2^* q u_2 = \left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{l} & O \\ \hline O & O \end{array}\right)$

(where k, l are nutural number with $0 \leq k, l \leq n$)

But since the following equalities hold :

$$k = \mathrm{Tr}\left(\begin{array}{c|c} \overbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}^{k} & 0 \\ \hline 0 & 0 \end{array}\right) = \mathrm{Tr}(u_1^* p u_1) = \mathrm{Tr}(p) = \mathrm{Tr}(v^* v)$$

$$= \mathrm{Tr}(v v^*) = \mathrm{Tr}(q) = \mathrm{Tr}(u_2^* q u_2)$$

$$= \mathrm{Tr}\left(\begin{array}{c|c} \overbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}^{l} & 0 \\ \hline 0 & 0 \end{array}\right) = l$$

(where $\mathrm{Tr}(a)$ is the trace of a),

One gets $u_1^* p u_1 = u_2^* q u_2$.

Thus $p = u_1 u_2^* q (u_1 u_2^*)^*$ $(u_1 u_2^* \in \mathcal{U}(M_n(\mathbb{C})))$.

The first part is proved.

proof of ②

Assume that there exists $u \in \mathcal{U}(M_n(\mathbb{C}))$ such that

$$q = u p u^*.$$

By Lemma 1, u is contained in $\mathcal{U}_0(M_n(\mathbb{C}))$.

So, we have $p \underset{h}{\sim} q$ in $\mathcal{P}(M_n(\mathbb{C}))$ by Lemma 2.

This completes the proof. ■

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