Homework 6

Recall that *a vector field* is a function F defined on an open set $\Omega \subset \mathbb{R}^n$ and taking values in $\mathbb{R}^n$. Such a vector field F admits *a potential function* if there exists a differentiable function $\phi : \Omega \to \mathbb{R}$ such that $F = \nabla\phi$.

Exercise 1 *a) Let $\Omega \subset \mathbb{R}^2$ be open and let $F : \Omega \to \mathbb{R}^2$ be of class C^1 on Ω. Let us also set $F = (f_1, f_2)$. Show that if*

$$\partial_x f_2 \neq \partial_y f_1$$

then F does not admit a potential function of class C^2.

b) What would be a similar statement for a function $F : \Omega \to \mathbb{R}^3$ if Ω is an open subset of $\mathbb{R}^3$.

c) What about the n-dimensional case, and how many conditions have to be satisfied ?

Exercise 2 *Determine which of the following vector fields admit a potential function. If the function f admits a potential function, exhibit it.*

(i) $f(x, y) = (1/x, x\mathrm{e}^{xy})$,

(ii) $f(x, y) = \big(\sin(xy), \cos(xy)\big)$,

(iii) $f(x, y) = \big(\mathrm{e}^{xy}, \mathrm{e}^{x+y}\big)$,

(iv) $f(x, y, z) = \big(y \sin(z), x \sin(z), xy \cos(z)\big)$.

Exercise 3 *a) For $(x, y) \in \mathbb{R}^2$ we set $r := \sqrt{x^2 + y^2}$. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = \big(\frac{\mathrm{e}^r}{r}x, \frac{\mathrm{e}^r}{r}y\big)$. Show that F admits a potential function and compute it.*

b) More generally, show that if $F : \mathbb{R}^n \to \mathbb{R}^n$ has the form $F(X) = \frac{g'(\|X\|)}{\|X\|}X$ for any $X \in \mathbb{R}^n$ and some differentiable function $g : \mathbb{R}_+ \to \mathbb{R}$, then F always admits a potential function.

Exercise 4 *Consider the vector field $F : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}^2$ defined for $(x, y) \neq (0, 0)$ by*

$$F(x, y) = \Big(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\Big)$$

(i) Represent graphically this vector field (you can use polar coordinates),

(ii) Can you find a potential function for this vector field, and if so exhibit it.

Exercise 5 *Consider the following expressions $f(x, y)$ which depend on the two variables (x, y) and compute by two different methods the expression $[\partial_x f](x, y)$ and $[\partial_y f](x, y)$:*

(i) $f(x, y) = \int_1^x \mathrm{e}^{ty}\,\mathrm{d}t$,

(ii) $f(x, y) = \int_1^x \mathrm{e}^{y+t}\,\mathrm{d}t$.

Can you explain your findings and justify more precisely your computations ?