Homework 5

Exercise 1 (Spherical coordinates) *Consider the map* $\Phi : [0,\infty) \times [0, 2\pi) \times [0, \pi) \to \mathbb{R}^3$ *with*

$$\Phi(r, \theta, \varphi) := \big(r\cos(\theta)\sin(\varphi), r\sin(\theta)\sin(\varphi), r\cos(\varphi)\big).$$

Compute the Jacobian matrix corresponding to this function.

Exercise 2 *Let us consider* $f : \mathbb{R}^3 \to \mathbb{R}$ *and* $F : \mathbb{R}^3 \to \mathbb{R}^3$ *sufficiently differentiable on* $\mathbb{R}^3$*. One also sets*

$$\begin{aligned}
\mathrm{grad}(f) &\equiv \nabla f = {}^T\big(\partial_1 f, \partial_2 f, \partial_3 f\big)\\
\mathrm{div}(F) &\equiv \nabla \cdot F = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3\\
\mathrm{curl}(F) &\equiv \nabla \times F = {}^T\big(\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1\big)\\
\Delta f &= \partial_1^2 f + \partial_2^2 f + \partial_3^2 f.
\end{aligned}$$

Show then the following relations:

(i) $\mathrm{div}(fF) = f\mathrm{div}(F) + \mathrm{grad}(f) \cdot F$,

(ii) $\mathrm{div}\big(\mathrm{curl}(F)\big) = 0$,

(iii) $\mathrm{curl}\big(\mathrm{grad}(f)\big) = 0$,

(iv) $\mathrm{curl}\big(\mathrm{curl}(F)\big) = \mathrm{grad}\big(\mathrm{div}(F)\big) - \Delta F$.

In the following exercises we call *a vector field* a function f defined on an open set $\Omega \subset \mathbb{R}^n$ and taking values in $\mathbb{R}^n$. In other words, for $\Omega \subset \mathbb{R}^n$ open, a function $f : \Omega \to \mathbb{R}^d$ is a vector field if $d = n$. For such a vector field f if there exists a differentiable function $\phi : \Omega \to \mathbb{R}$ such that $f = \nabla\phi$ we say that ϕ is a *potential function for* f.

Exercise 3 *Provide a picture for the following vector fields:*

(i) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $\quad f(x, y) = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$,

(ii) $g : \mathbb{R}^2 \to \mathbb{R}^2$, $\quad g(x, y) = x\,E_1 + y\,E_2$,

(iii) $\nabla k : \mathbb{R}^2 \to \mathbb{R}^2$ *for* $k : \mathbb{R}^2 \to \mathbb{R}$ *given by* $k(x, y) = \frac{1}{1+x^2+y^2}$.

Exercise 4 *For the following functions, does it exist a potential function ?*

(i) $f(x, y) = (y, x)$,

(ii) $f(x, y) = (3x^2y + 2x + y^3, x^3 + 3xy^2 - 2y)$,

(iii) $f(x, y) = \big(\cos(x), \sin(y)\big)$,

(iv) $f(x, y, z) = \big(x^2 - yz, y^2 - zx, z^2 - xy\big)$.