Nagoya University, G30 program Spring 2016

Mathematics Tutorial IIa (calculus) Instructor : Serge Richard

---

Homework 3

Exercise 1 *Consider the function* $f:\mathbb{R}^2\to\mathbb{R}$ *defined by*

$$f(x,y) := \begin{cases} \frac{x^3}{x^2+y^2} & \text{if} \quad (x,y)\neq(0,0) \\ 0 & \text{if} \quad (x,y)=(0,0) \end{cases}$$

1. *By using the polar coordinates, study the continuity of* f *at* $(0,0)$*,*
2. *Show that* f *is differentiable with continuous partial derivatives on* $\mathbb{R}^2\setminus\{(0,0)\}$*,*
3. *Show that* f *admits in* $(0,0)$ *derivatives in all directions,*
4. *Show that* f *is not differentiable at* $(0,0)$*.*

Exercise 2 *(i) Compute the Taylor expansion around* $(0,0)$ *and up to the second order of the function*

$$\mathbb{R}^2 \ni (x,y)\mapsto \mathrm{e}^{x^2+xy+y^2}\in\mathbb{R}.$$

(ii) Compute the Taylor expansion around $(0,0)$ *up to the third order of the function*

$$\mathbb{R}^2 \ni (x,y)\mapsto \mathrm{e}^{x+y}\in\mathbb{R}.$$

By fixing then $x=y=1/2$ *in the polynomial you have obtained, what can you say about the number* e *?*

Exercise 3 (The chain rule) *Let* Ω *be an open set in* $\mathbb{R}^n$ *and let* $f:\Omega\to\mathbb{R}$ *be differentiable. Let* (a,b) *be an open interval in* $\mathbb{R}$*, and consider* $\varphi:(a,b)\to\Omega$ *be a parametric curve which is differentiable. Show that the following equality holds:*

$$f\big(\varphi(t)\big)' := \frac{\mathrm{d}f\big(\varphi(t)\big)}{\mathrm{d}t} = [\nabla f](\varphi(t))\cdot\varphi'(t).$$

Exercise 4 (Geometrical interpretation of the gradient) *a) Consider the function* $f:\mathbb{R}^2\to\mathbb{R}$ *given by* $f(x,y)=x^2+4y^2$*.*

(i) Compute the gradient of f *at any point* (x,y)*,*

(ii) For $k>0$*, describe the* k*-level set* L_k*, and for this level set express* y *as a function of* x *and* k*,*

(iii) For any $(x,y)\in\mathbb{R}^2$ *such that* $f(x,y)=k$*, show that the gradient of* f *at* (x,y) *is orthogonal to the curve described by* L_k*.*

b) More generally, let $f:\mathbb{R}^n\to\mathbb{R}$ *be a function of class* C^1 *and let* $X\in\mathbb{R}^n$ *such that* $[\nabla f](X)\neq\mathbf{0}$*. Let* $k\in\mathbb{R}$ *be given by* $k:=f(X)$ *and consider the* k*-level set* L_k*. This* k*-level set can be considered (at least locally) as a surface of dimension* $n-1$ *in* $\mathbb{R}^n$*. Show that* $[\nabla f](X)$ *is perpendicular to the surface* L_k*. For that purpose, we can consider any parametric curve* $\varphi:(-1,1)\to L_k$ *with* $\varphi(0)=X$ *and show that* $[\nabla f](X)$ *is perpendicular to it at the point* X*.*