Homework 2

Exercise 1 *Let $\Omega \subset \mathbb{R}^2$ and consider the functions $f_i : \Omega \to \mathbb{R}$ defined for $(x,y) \in \Omega$ by*

$$a)\ f_1(x,y) = xy, \quad b)\ f_2(x,y) = (x+1)(y+3) \quad c)\ f_3(x,y) = \frac{xy}{x^2+y^2} \quad d)\ f_4(x,y) = \frac{x+y}{x-y}.$$

1. *Determine the maximal domain Ω on which these functions are well defined,*
2. *Sketch the k-level sets for these functions.*

Exercise 2 *Consider the following functions defined on $\mathbb{R}^2 \setminus \{(0,0)\}$*

$$a)\ f_1(x,y) = \frac{x^2-y^2}{x^2+y^2}, \quad b)\ f_2(x,y) = \frac{xy}{x^2+y^2}, \quad c)\ f_3(x,y) = \frac{1}{x^2+y^2+1}.$$

For each of them compute the limits $\lim_{x\to 0}\big(\lim_{y\to 0} f_i(x,y)\big)$, $\lim_{y\to 0}\big(\lim_{x\to 0} f_i(x,y)\big)$, and $\lim_{(x,y)\to(0,0)} f_i(x,y)$. Discuss your result.

Exercise 3 *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) := \begin{cases} \frac{x^2 y}{x^4 - 2x^2 y + 3y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

1. *Study the limit $(x,y) \to (0,0)$ along the path of equation $y = mx$ for any $m \in \mathbb{R}$,*
2. *Study the limit $(x,y) \to (0,0)$ along the path of equation $y = x^2$,*
3. *Show that f is not continuous at $(0,0)$.*

Exercise 4 *Compute the partial derivatives of the functions of Exercise 1 on their respective domain. Compute also the partial derivatives of the following functions:*

(i) $f : \mathbb{R}_+ \times \mathbb{R} \ni (x,y) \mapsto f(x,y) = x^y \in \mathbb{R}$,

(ii) $g : (\mathbb{R}_+)^3 \to \mathbb{R}$ *given by* $g(x,y,z) = x^3 y^2 + \sin(xz) - \ln(xyz)$.

Let us recall the polar coordinates $(r,\theta) \in [0,\infty) \times [0,2\pi)$ which allow one to give an alternative description of all the points of $\mathbb{R}^2$. The relations between the usual coordinates $(x,y) \in \mathbb{R} \times \mathbb{R}$ are

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta).$$

Exercise 5 *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) := \begin{cases} \frac{x^3}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

1. *By using the polar coordinates, study the continuity of f at $(0,0)$,*
2. *Show that f is differentiable with continuous partial derivatives on $\mathbb{R}^2 \setminus \{(0,0)\}$,*
3. *Show that f admits in $(0,0)$ derivatives in all directions,*
4. *Show that f is not differentiable at $(0,0)$.*