
Homework 12

Exercise 1 Consider the parametrization of the sphere of radius $r > 0$ given by $f : [0, 2\pi) \times [0, \pi) \rightarrow \mathbb{R}^3$ with

$$f(\theta, \varphi) := \begin{pmatrix} r \cos(\theta) \sin(\varphi) \\ r \sin(\theta) \sin(\varphi) \\ r \cos(\varphi) \end{pmatrix}.$$

Compute the vectors $[\partial_1 f](\theta, \varphi)$, $[\partial_2 f](\theta, \varphi)$, and the vector normal to the sphere at the point $f(\theta, \varphi)$.

Exercise 2 Let $g : [0, 1] \rightarrow \mathbb{R}_+$ of class C^1 and consider the surface of revolution defined by

$$f : [0, 1] \times [0, 2\pi) \ni (x, \theta) \mapsto \begin{pmatrix} x \\ g(x) \cos(\theta) \\ g(x) \sin(\theta) \end{pmatrix} \in \mathbb{R}^3.$$

Compute the area of this surface. You can compare your result with what has been obtained in Homework 8, Exercise 3 of Calculus I.

Exercise 3 Let $\Omega \subset \mathbb{R}^2$ be open and let $g : \Omega \rightarrow \mathbb{R}$ be of class C^1 . We consider the surface of \mathbb{R}^3 parameterized by the function $f : \Omega \rightarrow \mathbb{R}^3$ defined by $f(x, y) = {}^T(x, y, g(x, y))$. Compute the area of the surface $f(\Omega)$.

Exercise 4 In the setting of the previous exercise, compute the area of the surface defined by

(i) Ω is the disc of radius 1 centered at $(0, 0) \in \mathbb{R}^2$ and $g(x, y) = x^2 + y^2$,

(ii) Ω is the disc of radius 1 centered at $(0, 0) \in \mathbb{R}^2$ and $g(x, y) = xy$,

Exercise 5 Consider the upper half-sphere S in \mathbb{R}^3 centered in $(0, 0, 0)$ and of radius R , and let $\Psi : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $\Psi(x, y, z) = x^2 + y^2$. Compute the integral $\iint_S \Psi \, d\sigma$ of Ψ on the upper half-sphere. Same question for Ψ defined by $\Psi(x, y, z) = (x^2 + y^2)z$. The result of Exercise 1 can be used.

Exercise 6 Consider the vector field Ψ in \mathbb{R}^3 defined by $\Psi(x, y, z) = (x, y, 0)$. Compute the flux of this vector field through the sphere in \mathbb{R}^3 centered at $(0, 0, 0)$ and of radius r . The result of Exercise 1 can be used.