Nagoya University, G30 program Spring 2015

Mathematics Tutorial II (linear algebra part) Instructor : Serge Richard

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Homework 7

Exercise 1 *For an arbitrary field* $\mathbb{F}$ *let* $\mathcal{A} = (a_{ij}) \in M_n(\mathbb{F})$ *and recall the formula:*

$$\begin{aligned} \mathsf{Det}(\mathcal{A}) &= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \, \mathsf{Det}\big(\mathcal{A}(i,j)\big) \qquad \text{for any fixed } j \in \{1,\dots,n\} \\ &= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \, \mathsf{Det}\big(\mathcal{A}(i,j)\big) \qquad \text{for any fixed } i \in \{1,\dots,n\}. \end{aligned}$$

Show that $\mathsf{Det}(\mathbf{1}_n) = 1$ *for any* n*. For* $n = 2$*, show that*

(i) the determinant is linear as a function of the columns of $\mathcal{A}$*,*

(ii) the determinant is alternating as a function of the columns of $\mathcal{A}$*.*

Can you do it for $n = 3$*? For arbitrary* n *(a proof by induction over the dimension* n *is recommended).*

Exercise 2 *Let* $\mathcal{A} = (a_{jk}) \in M_n(\mathbb{R})$ *be an upper triangular matrix. Compute* $\mathsf{Det}(\mathcal{A})$*.*

Exercise 3 *For* $r \in \{1,\dots,m\}$ *and* $s \in \{1,\dots,m\}$*, let* $I_{rs} \in M_m(\mathbb{F})$ *be the matrix whose* rs*-component is* 1 *and all the other ones are equal to* 0*. For* $c \neq 0$*, consider the following 3 types of elementary matrices :*

1. $\mathbf{1}_m - I_{rr} + cI_{rr}$*, the matrix obtained from the identity matrix by multiplying the* r*-th diagonal component by* c*,*
2. *For* $r \neq s$*,* $(\mathbf{1}_m + I_{rs} + I_{sr} - I_{rr} - I_{ss})$*, the matrix obtained from the identity matrix by interchanging the* r*-th row with the* s*-th row,*
3. *For* $r \neq s$*,* $(\mathbf{1}_m + cI_{rs})$*, the matrix having the* rs*-th component equal to* c*, all other components* 0 *except the diagonal components which are equal to* 1*.*

Compute the determinant of these elementary matrices.

Exercise 4 *Compute the determinant of the following matrices:*

$$a)\begin{pmatrix} 4 & -1 & 1 \\ 2 & 0 & 0 \\ 1 & 5 & 7 \end{pmatrix}, \quad b)\begin{pmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 8 \end{pmatrix} \quad c)\begin{pmatrix} 3 & 1 & 1 \\ 2 & 5 & 5 \\ 8 & 7 & 7 \end{pmatrix} \quad d)\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & 1 & 0 \\ 3 & 0 & 0 & 5 \end{pmatrix}$$