Homework 2

Exercise 1 *Let* $\mathrm{P} : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ *be the map defined for any* $\mathcal{A} \in M_n(\mathbb{R})$ *by*

$$\mathrm{P}(\mathcal{A}) = \frac{1}{2}(\mathcal{A} + {}^t\!\mathcal{A}).$$

1. *Show that* P *is a linear map.*
2. *Show that the kernel of* P *consists in the vector space of all skew-symmetric matrices.*
3. *Show that the range of* P *consists in the vector space of all symmetric matrices.*
4. *What is the dimension of the vector space of all symmetric matrices, and the dimension of the vector space of all skew-symmetric matrices ?*

Exercise 2 *Let* $\mathcal{A}$ *be the matrix given by* $\mathcal{A} = \begin{pmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{pmatrix}$ *and consider the linear map* $\mathrm{L}_\mathcal{A} : \mathbb{R}^4 \to \mathbb{R}^3$ *defined by* $\mathrm{L}_\mathcal{A} X = \mathcal{A}X$ *for all* $X \in \mathbb{R}^4$.

1. *Determine the rank of* $\mathcal{A}$ *and the dimension of the range of* $\mathrm{L}_\mathcal{A}$.
2. *Deduce the dimension of the kernel of* $\mathrm{L}_\mathcal{A}$*, and exhibit a basis for the kernel of* $\mathrm{L}_\mathcal{A}$.
3. *Find the set of all solutions of the equation* $\mathcal{A}X = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$.

Exercise 3 *Let* $\mathrm{F} : \mathbb{R}^3 \to \mathbb{R}^2$ *be the map indicated below. What is the matrix associated with* F *in the canonical bases of* $\mathbb{R}^3$ *and* $\mathbb{R}^2$ *?*

$$a)\quad \mathrm{F}(E_1) = \begin{pmatrix} 1 \\ -3 \end{pmatrix},\ \mathrm{F}(E_2) = \begin{pmatrix} -4 \\ 2 \end{pmatrix},\ \mathrm{F}(E_3) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

and

$$b)\quad \mathrm{F}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 - 2x_2 + x_3 \\ 4x_1 - x_2 + 5x_3 \end{pmatrix}.$$

Exercise 4 *Let* $\mathrm{L} : \mathbb{R}^3 \to \mathbb{R}^3$ *be a linear map which associated matrix has the form* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ *with respect to the canonical basis of* $\mathbb{R}^3$. *What is the matrix associated with* L *in the basis generated by the three vectors* $V_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$, $V_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$, $V_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

Exercise 5 *Let* U, V, W *be vector spaces over a field , and let* $\mathrm{G} : U \to V$, $\mathrm{G}' : U \to V$, $\mathrm{H} : V \to W$ *and* $\mathrm{H}' : V \to W$ *be linear maps. Show that*

(i) $\mathrm{H} \circ \mathrm{G} : U \to W$ *is a linear map,*

(ii) $(\mathrm{H} + \mathrm{H}') \circ \mathrm{G} = \mathrm{H} \circ \mathrm{G} + \mathrm{H}' \circ \mathrm{G}$,

(iii) $\mathrm{H} \circ (\mathrm{G} + \mathrm{G}') = \mathrm{H} \circ \mathrm{G} + \mathrm{H} \circ \mathrm{G}'$,

(iv) $(\lambda \mathrm{H}) \circ \mathrm{G} = \mathrm{H} \circ (\lambda \mathrm{G}) = \lambda(\mathrm{H} \circ \mathrm{G})$, *for all* $\lambda \in$.