**Nagoya University, G30 program** Spring 2015

Mathematics Tutorial II (linear algebra part) Instructor : Serge Richard

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**Homework 12**

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**Exercise 1** *For a symmetric matrix $\mathcal{A} \in M_n(\mathbb{R})$, one says that $\mathcal{A}$ is* positive definite *if $\langle \mathcal{A}X, X\rangle > 0$ for any $X \in \mathbb{R}^n$ with $X \neq 0$. In fact, this is precisely the condition which makes the bilinear map $\mathrm{F}_\mathcal{A}$ define a scalar product, see Exercise 3 in Homework 6. If $\mathcal{A}$ is symmetric and positive definite, show that*

*(i) All eigenvalues of $\mathrm{L}_\mathcal{A}$ are strictly positive,*

*(ii) $\mathcal{A}^2$ is symmetric and positive definite,*

*(iii) $\mathcal{A}^{-1}$ is symmetric and positive definite.*

**Exercise 2** *Let $\mathcal{A} \in M_n(\mathbb{R})$ be symmetric. Show that there exists $\mathcal{B} \in M_n(\mathbb{R})$ such that $\mathcal{B}^3 = \mathcal{A}$.*

**Exercise 3** *Let $\mathcal{A} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$, and consider the associated linear map $\mathrm{L}_\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$. Compute $\mathcal{A}^n$ for $n = 2$, $n = 3$, $n = 25$ and $n = \infty$. You are allowed to use the result of Homework 10, Exercise 6.*

**Exercise 4** *Let $\mathcal{A} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, and consider the associated linear map $\mathrm{L}_\mathcal{A} : \mathbb{C}^2 \to \mathbb{C}^2$. Determine the eigenvalues of $\mathrm{L}_\mathcal{A}$ and the corresponding eigenspaces.*

**Exercise 5** *Let $\mathcal{A} \in M_n(\mathbb{R})$ and assume that $\mathrm{L}_\mathcal{A}$ has $n$ eigenvalues $\lambda_1, \dots, \lambda_n$. Then, show the following equalities:*

*(i)* $\mathsf{Det}(\mathcal{A}) = \lambda_1\lambda_2 \dots \lambda_n$ *(product of the eigenvalues)*

*(ii)* $\mathsf{Tr}(\mathcal{A}) = \lambda_1 + \lambda_2 + \dots + \lambda_n$ *(sum of the eigenvalues)*