Nagoya University, G30 program Spring 2015

Mathematics Tutorial II (linear algebra part) Instructor : Serge Richard

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Homework 11

Exercise 1 *For $z, z' \in \mathbb{C}$, is it true that $\Re(zz') = \Re(z)\Re(z')$, with $\Re(z)$ the real part of z ?*

Exercise 2 *Compute the following expressions:*

$$(i)\ (1+3i)+(1-2i) \qquad (ii)\ (2+3i)(1-2i) \qquad (iii)\ i^{-1} \qquad (iv)\ (1+i)^{-1} \qquad (v)\ \sqrt{i}.$$

Recall that for any complex number $z = x + iy$ we define $r := |z| \equiv \sqrt{x^2 + y^2}$ and set

$$z = r\big(\cos(\theta) + i\sin(\theta)\big)$$

with $x = r\cos(\theta)$ and $y = r\sin(\theta)$. This is called the polar coordinate representation *of the complex number z. The number r is called the* norm *or the* modulus *of z, and θ its* argument*, i.e. $\theta = \arg(z)$.*

Exercise 3 *For any $z_1, z_2 \in \mathbb{C}$, show that $|z_1 z_2| = |z_1||z_2|$ and that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.*

Exercise 4 *Deduce from the previous exercise* de Moivre's formula*: for $z = r\big(\cos(\theta) + i\sin(\theta)\big)$ and for $n \in \mathbb{N}$*

$$z^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big).$$

Exercise 5 *Deduce that for any complex number $z = r\big(\cos(\theta) + i\sin(\theta)\big)$, the n-th roots of z are given by*

$$z_j := \sqrt[n]{r}\left[\cos\Big(\frac{\theta + 2\pi j}{n}\Big) + i\sin\Big(\frac{\theta + 2\pi j}{n}\Big)\right]$$

for $j \in \{0, 1, \dots, n-1\}$.

Recall now that for any $z = x + iy \in \mathbb{C}$, we call $\overline{z} := x - iy$ the complex conjugate *of z.*

Exercise 6 *Show the following properties:*

$$(i)\ \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \qquad (ii)\ \overline{z_1 z_2} = \overline{z_1}\ \overline{z_2} \qquad (iii)\ z\,\overline{z} = |z|^2$$

and also $z^{-1} = \overline{z}/|z|^2$ whenever $z \neq 0$, and that $\Re(z) = (z + \overline{z})/2$ and $\Im(z) = (z - \overline{z})/(2i)$, where $\Im(z)$ is the imaginary part of z.

Exercise 7 *Show also that $|\overline{z}| = |z|$ and that $\arg(\overline{z}) = -\arg(z)$.*

For $z = x + iy$, let us set

$$e^z = e^{x+iy} := e^x\big(\cos(y) + i\sin(y)\big).$$

Exercise 8 *Show the following properties:*

1. $e^{z_1 + z_2} = e^{z_1}\, e^{z_2}$ *for any $z_1, z_2 \in \mathbb{C}$,*
2. e^z *is never equal to* 0,
3. $|e^{x+iy}| = e^x$,
4. $e^{i\pi} = -1$ *(Euler's identity, and "one of the most beautiful formula in mathematics").*