Nagoya University, G30 program Spring 2015

Mathematics Tutorial II (linear algebra part) Instructor : Serge Richard

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Homework 10

Exercise 1 *For any* $\mathcal{A} \in M_2(\mathbb{R})$*, show the following equality*

$$P_{\mathcal{A}}(\lambda) = \lambda^2 - \lambda\,\mathsf{Tr}(\mathcal{A}) + \mathsf{Det}(\mathcal{A}).$$

Exercise 2 *Let* $\mathcal{A} \in M_n(\mathbb{R})$ *and assume that* $\mathcal{A}$ *has* n *eigenvalues* $\lambda_1, \dots, \lambda_n$*. Then, show the following equalities:*

(i) $\mathsf{Det}(\mathcal{A}) = \lambda_1\lambda_2\dots\lambda_n$ *(product of the eigenvalues)*

(ii) $\mathsf{Tr}(\mathcal{A}) = \lambda_1 + \lambda_2 + \dots + \lambda_n$ *(sum of the eigenvalues)*

Exercise 3 *Let* $\mathcal{A} \in M_n(\mathbb{R})$ *and consider the linear maps* $\mathrm{L}_{\mathcal{A}}$ *and* $\mathrm{L}_{{}^t\!\mathcal{A}}$*. Show that these linear maps have the same eigenvalues.*

Exercise 4 *Show that if* $\mathcal{A} \in M_n(\mathbb{R})$ *is orthogonal (i.e.* ${}^t\!\mathcal{A} = \mathcal{A}^{-1}$*), then the (real) eigenvalues of* $\mathrm{L}_{\mathcal{A}}$ *can only be* 1 *or* -1*.*

Exercise 5 *Let* $\mathcal{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$*, and consider the associated linear map* $\mathrm{L}_{\mathcal{A}} : \mathbb{R}^3 \to \mathbb{R}^3$*. Determine the eigenvalues of* $\mathrm{L}_{\mathcal{A}}$ *and the corresponding eigenspaces.*

Exercise 6 *Let* $\mathcal{A} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$*, and consider the associated linear map* $\mathrm{L}_{\mathcal{A}} : \mathbb{R}^2 \to \mathbb{R}^2$*. Determine the eigenvalues of* $\mathrm{L}_{\mathcal{A}}$ *and the corresponding eigenspaces. Consider then the matrix* $\mathcal{B} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ *and compute the product* $\mathcal{B}^{-1}\mathcal{A}\mathcal{B}$*. What do you observe, and how do you understand your result ?*

Exercise 7 *Let* $\mathcal{A} = \begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$*, and consider the associated linear map* $\mathrm{L}_{\mathcal{A}} : \mathbb{R}^2 \to \mathbb{R}^2$*. Determine the eigenvalues of* $\mathrm{L}_{\mathcal{A}}$ *and the corresponding eigenspaces.*