Chapter 3

K_0-group for a unital C^*-algebra

In this chapter, we associate with each unital C^*-algebra an Abelian group. This group will be constructed from equivalence classes of projections. The K_0-group for non-unital C^*-algebra will be described in the next Chapter.

3.1 Semigroups of projections

Let us start by introducing a semigroup of projections in a C^*-algebra, with or without a unit. For that purpose, let $\mathcal{C}$ be an arbitrary C^*-algebra and set for $n \in \mathbb{N}^*$

$$\mathcal{P}_n(\mathcal{C}) := \mathcal{P}\big(M_n(\mathcal{C})\big) \qquad \text{and} \qquad \mathcal{P}_\infty(\mathcal{C}) := \bigcup_{n=1}^{\infty} \mathcal{P}_n(\mathcal{C}).$$

One can then define the relation $\sim_0$ on $\mathcal{P}_\infty(\mathcal{C})$, namely for two elements $p, q \in \mathcal{P}_\infty(\mathcal{C})$ one writes $p \sim_0 q$ if there exits $v \in M_{m,n}(\mathcal{C})$ such that $p = v^*v \in \mathcal{P}_n(\mathcal{C})$ and $q = vv^* \in \mathcal{P}_m(\mathcal{C})$. Clearly, $M_{m,n}(\mathcal{C})$ denotes the set of $m \times n$ matrices with entries in $\mathcal{C}$, and the adjoint v^* of $v \in M_{m,n}(\mathcal{C})$ is obtained by taking the transpose of the matrix, and then the adjoint of each entry.

One easily observes that the relation $\sim_0$ is an equivalence relation on $\mathcal{P}_\infty(\mathcal{C})$. It combines both the Murray-von Neumann equivalence relation $\sim$ and and the identification of projections in different sized matrix algebras over $\mathcal{C}$. For example, if $p, q \in \mathcal{P}_n(\mathcal{C})$ then $p \sim_0 q$ if and only if $p \sim q$.

We also define a binary operation $\oplus$ on $\mathcal{P}_\infty(\mathcal{C})$ by

$$p \oplus q = \mathrm{diag}(p, q) := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix},$$

so that $p \oplus q$ belongs to $\mathcal{P}_{m+n}(\mathcal{C})$ whenever $p \in \mathcal{P}_n(\mathcal{C})$ and $q \in \mathcal{P}_m(\mathcal{C})$. We can now derive some of the properties of $\sim_0$.

Proposition 3.1.1. *Let $\mathcal{C}$ be a C^*-algebra, and let p, q, r, p', q' be elements of $\mathcal{P}_\infty(\mathcal{C})$. Then:*

(i) $p \sim_0 p \oplus 0_n$ *for any natural number* n*, where* 0_n *denotes the* 0*-element of* $M_n(\mathcal{C})$*,*

(ii) *If* $p \sim_0 p'$ *and* $q \sim_0 q'$*, then* $p \oplus q \sim_0 p' \oplus q'$*,*

(iii) $p \oplus q \sim_0 q \oplus p$*,*

(iv) *If* $p, q \in \mathcal{P}_n(\mathcal{C})$ *such that* $pq = 0$*, then* $p + q \in \mathcal{P}_n(\mathcal{C})$ *and* $p + q \sim_0 p \oplus q$*,*

(v) $(p \oplus q) \oplus r = p \oplus (q \oplus r)$*.*

Proof. i) Let m, n be integers, and let $p \in \mathcal{P}_m(\mathcal{C})$. One then sets $v := \binom{p}{0} \in M_{m+n,m}(\mathcal{C})$, and one gets $p = v^*v$ and $vv^* = p \oplus 0_n$.

ii) Let v, w such that $p = v^*v$, $p' = vv^*$, $q = w^*w$ and $q' = ww^*$, and set $u := \mathrm{diag}(v, w)$. Then $p \oplus q = u^*u$ and $p' \oplus q' = uu^*$.

iii) Assume $p \in \mathcal{P}_n(\mathcal{C})$ and $q \in \mathcal{P}_m(\mathcal{C})$, and set $v := \begin{pmatrix} 0_{n,m} & q \\ p & 0_{m,n} \end{pmatrix}$, with $0_{k,l}$ the 0-matrix of size $k \times l$. Then one gets $p \oplus q = v^*v$ and $q \oplus p = vv^*$.

iv) If $pq = 0$ it is easily observe that $p + q$ is itself a projection. Then, if one sets $v := \binom{p}{q} \in M_{2n,n}(\mathcal{C})$, one gets $p + q = v^*v$ and $p \oplus q = vv^*$.

v) This last statement is trivial. □

Definition 3.1.2. *For any* C^**-algebra* $\mathcal{C}$*, one sets*

$$\mathcal{D}(\mathcal{C}) := \mathcal{P}_\infty(\mathcal{C}) / \sim_0$$

which corresponds to the equivalent classes of elements of $\mathcal{P}_\infty(\mathcal{C})$ *modulo the equivalence relation* $\sim_0$*. For any* $p \in \mathcal{P}_\infty(\mathcal{C})$ *one writes* $[p]_\mathcal{D} \in \mathcal{D}(\mathcal{C})$ *for the equivalent class containing* p*. The set* $\mathcal{D}(\mathcal{C})$ *is endowed with a binary operation defined for any* $p, q \in \mathcal{P}_\infty(\mathcal{C})$ *by*

$$[p]_\mathcal{D} + [q]_\mathcal{D} = [p \oplus q]_\mathcal{D}. \tag{3.1}$$

Because of the previous proposition, one directly infers the following result:

Lemma 3.1.3. *The pair* $\big(\mathcal{D}(\mathcal{C}), +\big)$ *defines an Abelian semigroup.*

We end this section with two exercises dealing with projections.

Exercise 3.1.4. *Let* $\mathrm{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ *denote the usual trace on square matrices, and let* $p, q \in \mathcal{P}\big(M_n(\mathbb{C})\big)$*. Show that the following statements are equivalent:*

(i) $p \sim q$*,*

(i) $\mathrm{tr}(p) = \mathrm{tr}(q)$*,*

(i) $\dim\big(p(\mathbb{C}^n)\big) = \dim\big(q(\mathbb{C}^n)\big)$*.*

Use this to show that $\mathcal{D}(\mathbb{C}) \cong \mathbb{Z}_+ \equiv \{0, 1, 2, \dots\}$ *when* $\mathbb{Z}_+$ *is equipped with the usual addition.*

Exercise 3.1.5. *Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space, and let p, q be projections in $\mathcal{B}(\mathcal{H})$.*

(i) Show that $p \sim q$ if and only if $\dim\big(p(\mathcal{H})\big) = \dim\big(q(\mathcal{H})\big)$,

(ii) Show that $p \sim_u q$ if and only if $\dim\big(p(\mathcal{H})\big) = \dim\big(q(\mathcal{H})\big)$ and $\dim\big(p(\mathcal{H})^\perp\big) = \dim\big(q(\mathcal{H})^\perp\big)$,

(iii) Infer that $\mathcal{D}\big(\mathcal{B}(\mathcal{H})\big) \cong \mathbb{Z}_+ \cup \{\infty\} \equiv \{0, 1, 2, \dots, \infty\}$, where the usual addition on $\mathbb{Z}_+$ is considered together with the addition $n + \infty = \infty + n = \infty$ for all $n \in \mathbb{Z}_+ \cup \{\infty\}$.

3.2 The K_0-group

In this section we construct the K_0-group associated with a unital C^*-algebra $\mathcal{C}$. This group is defined in terms of the Grothendieck construction applied to the Abelian semigroup $\big(\mathcal{D}(\mathcal{C}), +\big)$. We first recall this construction in an abstract setting.

Let $(\mathcal{D}, +)$ be an Abelian semigroup, and define on $\mathcal{D}\times\mathcal{D}$ the relation $\sim$ by $(x_1, y_1) \sim (x_2, y_2)$ if there exists $z \in \mathcal{D}$ such that $x_1 + y_2 + z = x_2 + y_1 + z$. This relation is clearly reflexive and symmetric. For the transitivity, suppose that $(x_1, y_1) \sim (x_2, y_2)$ and that $(x_2, y_2) \sim (x_3, y_3)$. This means that there exist $z, w \in \mathcal{D}$ such that

$$x_1 + y_2 + z = x_2 + y_1 + z \qquad \text{and} \qquad x_2 + y_3 + w = x_3 + y_2 + w.$$

It then follows that

$$x_1 + y_3 + (y_2 + z + w) = x_2 + y_1 + z + y_3 + w = x_3 + y_1 + (y_2 + z + w)$$

so that $(x_1, y_1) \sim (x_3, y_3)$. As a consequence, $\sim$ defines an equivalence relation on $\mathcal{D}\times\mathcal{D}$. The equivalence class containing (x, y) is denoted by $\langle x, y\rangle$, and we set $\mathcal{G}(\mathcal{D})$ for the quotient $\mathcal{D} \times \mathcal{D}/\sim$. Then, the operation

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle$$

endows $\mathcal{G}(\mathcal{D})$ with the structure of an Abelian group. Indeed, the inverse $-\langle x, y\rangle$ of $\langle x, y\rangle$ is given by $\langle y, x\rangle$, and $\langle x, x\rangle = 0$, for any $x, y \in \mathcal{D}$. The pair $\big(\mathcal{G}(\mathcal{D}), +\big)$ is called *the Grothendieck group*.

For any fixed $y \in \mathcal{D}$, let us also define the map

$$\gamma_{\mathcal{D}} : \mathcal{D} \ni x \mapsto \gamma_{\mathcal{D}}(x) := \langle x + y, y\rangle \in \mathcal{G}(\mathcal{D}),$$

and observe that this map does not depend on the choice of any specific $y \in \mathcal{D}$. Indeed, one easily observes that $(x+y, y)$ and $(x+y', y')$ define the same equivalence class since $(x + y) + y' = (x + y') + y$. The map $\gamma_{\mathcal{D}}$ is called *the Grothendieck map*.

Finally, one says that the semigroup $(\mathcal{D}, +)$ has the *cancellation property* if whenever the equality $x+z = y+z$ holds, it follows that $x = y$. Let us now gather some additional information on this construction in the following proposition.

Proposition 3.2.1. *Let $(\mathcal{D},+)$ be an Abelian semigroup, and let $\big(\mathcal{G}(\mathcal{D}),+\big)$ and $\gamma_{\mathcal{D}}$ be the corresponding Grothendieck group and Grothendieck map. Then:*

(i) Universal property: *If H is an Abelian group and if $\varphi : \mathcal{D} \to H$ is an additive map, then there is one and only one group homomorphism $\psi : \mathcal{G}(\mathcal{D}) \to H$ making the diagram*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\ \varphi\ } & H \\ & {\scriptstyle\gamma_{\mathcal{D}}}\searrow & \uparrow{\scriptstyle\psi} \\ & & \mathcal{G}(\mathcal{D}) \end{array}$$

commutative,

(ii) Functoriality: *For every additive map $\varphi : \mathcal{D} \to \mathcal{D}'$ between semigroups there exists one and only one group morphism $\mathcal{G}(\varphi) : \mathcal{G}(\mathcal{D}) \to \mathcal{G}(\mathcal{D}')$ making the diagram*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\ \varphi\ } & \mathcal{D}' \\ {\scriptstyle\gamma_{\mathcal{D}}}\downarrow & & \downarrow{\scriptstyle\gamma_{\mathcal{D}'}} \\ \mathcal{G}(\mathcal{D}) & \xrightarrow[\mathcal{G}(\varphi)]{} & \mathcal{G}(\mathcal{D}') \end{array}$$

commutative,

(iii) $\mathcal{G}(\mathcal{D}) = \{\gamma_{\mathcal{D}}(x) - \gamma_{\mathcal{D}}(y) \mid x, y \in \mathcal{D}\}$,

(iv) For any $x, y \in \mathcal{D}$ one has $\gamma_{\mathcal{D}}(x) = \gamma_{\mathcal{D}}(y)$ if and only if $x + z = y + z$ for some $z \in \mathcal{D}$,

(v) The Grothendieck map $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathcal{G}(\mathcal{D})$ is injective if and only if $(\mathcal{D},+)$ has the cancellation property,

(vi) Let $(H,+)$ be an Abelian group, and let $\mathcal{D}$ be a non-empty subset of H. If $\mathcal{D}$ is closed under addition, then $(\mathcal{D},+)$ is an Abelian semigroup with the cancellation property. In addition, $\mathcal{G}(\mathcal{D})$ is isomorphic to the subgroup H_0 generated by $\mathcal{D}$, and $H_0 = \{x - y \mid x, y \in \mathcal{D}\}$.

The proofs of these statements can be found in [RLL00, Sec. 3.1.2]. Let us just mention the one of *(iii)*: Since each element of $\mathcal{G}(\mathcal{D})$ has the form $\langle x, y\rangle$ for some $x, y \in \mathcal{D}$, it is sufficient to observe that

$$\langle x, y\rangle = \langle x + y, y\rangle - \langle x + y, x\rangle = \gamma_{\mathcal{D}}(x) - \gamma_{\mathcal{D}}(y).$$

We still illustrate the previous construction with two examples.

Examples 3.2.2. *(i) The Grothendieck group of the Abelian semigroup $(\mathbb{Z}_+,+)$ is isomorphic to $(\mathbb{Z},+)$. Note that $(\mathbb{Z}_+,+)$ has the cancellation property.*

(ii) The Grothendieck group of the Abelian semigroup $\big(\mathbb{Z}_+ \cup \{\infty\}, +\big)$ *is* $\{0\}$*. Note that* $\big(\mathbb{Z}_+ \cup \{\infty\}, +\big)$ *does not possess the cancellation property.*

We are now ready for the main definition of this chapter. Recall that for any C^*-algebra $\mathcal{C}$, the Abelian semigroup $\big(\mathcal{D}(\mathcal{C}), +\big)$ has been introduced in Definition 3.1.2, see also Lemma 3.1.3.

Definition 3.2.3. *Let* $\mathcal{C}$ *be a unital* C^**-algebra, and let* $\big(\mathcal{D}(\mathcal{C}), +\big)$ *be the corresponding Abelian semigroup. The Abelian group* $K_0(\mathcal{C})$ *is defined by*

$$K_0(\mathcal{C}) := \mathcal{G}\big(\mathcal{D}(\mathcal{C})\big).$$

One also set $[\cdot]_0 : \mathcal{P}_\infty(\mathcal{C}) \to K_0(\mathcal{C})$ *for any* $p \in \mathcal{P}_\infty(\mathcal{C})$ *by*

$$[p]_0 := \gamma\big([p]_\mathcal{D}\big)$$

with $\gamma : \mathcal{D}(\mathcal{C}) \to K_0(\mathcal{C})$ *the Grothendieck map.*

In the following two propositions, we provide a standard picture of the K_0-group for a unital C^*-algebra, and state some of its universal properties. Before them, we introduce one more equivalence relation on $\mathcal{P}_\infty(\mathcal{C})$, namely $p, q \in \mathcal{P}_\infty(\mathcal{C})$ are *stable equivalent*, written $p \sim_s q$, if there exists $r \in \mathcal{P}_\infty(\mathcal{C})$ such that $p \oplus r \sim_0 q \oplus r$. Note that if $\mathcal{C}$ is unital, then $p \sim_s q$ if and only if $p \oplus \mathbf{1}_n \sim_0 q \oplus \mathbf{1}_n$ for some $n \in \mathbb{N}$. Indeed, if $p \oplus r \sim_0 q \oplus r$ for some $r \in \mathcal{P}_n(\mathcal{C})$, then

$$p \oplus \mathbf{1}_n \sim_0 p \oplus r \oplus (\mathbf{1}_n - r) \sim_0 q \oplus r \oplus (\mathbf{1}_n - r) \sim_0 q \oplus \mathbf{1}_n,$$

where Proposition 3.1.1.(iv) has been used twice.

Proposition 3.2.4. *For any unital* C^**-algebra* $\mathcal{C}$ *one has*

$$K_0(\mathcal{C}) = \big\{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_\infty(\mathcal{C})\big\} = \big\{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_n(\mathcal{C}), n \in \mathbb{N}^*\big\}. \tag{3.2}$$

Moreover, one has

(i) $[p \oplus q]_0 = [p]_0 + [q]_0$ *for any projections* $p, q \in \mathcal{P}_\infty(\mathcal{C})$,

(ii) $[0_\mathcal{C}] = 0$, *where* $0_\mathcal{C}$ *stands for the zero element of* $\mathcal{C}$,

(iii) If $p, q \in \mathcal{P}_n(\mathcal{C})$ *for some* $n \in \mathbb{N}^*$ *and if* $p \sim_h q \in \mathcal{P}_n(\mathcal{C})$, *then* $[p]_0 = [q]_0$,

(iv) If p, q *are mutually orthogonal projections in* $\mathcal{P}_n(\mathcal{C})$, *then* $[p + q]_0 = [p]_0 + [q]_0$,

(v) For all $p, q \in \mathcal{P}_\infty(\mathcal{C})$, *then* $[p]_0 = [q]_0$ *if and only if* $p \sim_s q$.

Proof. The first equality in (3.2) follows from Proposition 3.2.1.(*iii*). Hence, if g is an element of $K_0(\mathcal{C})$ there exist $p' \in \mathcal{P}_k(\mathcal{C})$ and $q' \in \mathcal{P}_l(\mathcal{C})$ such that $g = [p']_0 - [q']_0$. Choose then n greater than k and l, and set $p = p' \oplus 0_{n-k}$ and $q := q \oplus 0_{n-l}$. Then $p, q \in \mathcal{P}_n(\mathcal{C})$ with $p \sim_0 p'$ and $q \sim_0 q'$ by Proposition 3.1.1.(*i*). It thus follows that $g = [p]_0 - [q]_0$.

i) One has by (3.1)

$$[p \oplus q]_0 = \gamma\big([p \oplus q]_{\mathcal{D}}\big) = \gamma\big([p]_{\mathcal{D}} + [q]_{\mathcal{D}}\big) = \gamma\big([p]_{\mathcal{D}}\big) + \gamma\big([q]_{\mathcal{D}}\big) = [p]_0 + [q]_0.$$

ii) Since $0_{\mathcal{C}} \oplus 0_{\mathcal{C}} \sim_0 0_{\mathcal{C}}$, point (*i*) yields that $[0_{\mathcal{C}}]_0 + [0_{\mathcal{C}}]_0 = [0_{\mathcal{C}}]_0$, which means that $[0_{\mathcal{C}}]_0 = 0$.

iii) This statement follows from the implications

$$p \sim_h q \Rightarrow p \sim q \Rightarrow p \sim_0 q \Leftrightarrow [p]_{\mathcal{D}} = [q]_{\mathcal{D}} \Rightarrow [p]_0 = [q]_0,$$

where the first two relations are defined only when p and q are in the same matrix algebra over $\mathcal{C}$, while the three other implications hold for any $p, q \in \mathcal{P}_\infty(\mathcal{C})$. Note that the first implication is due to Lemma 2.2.9.

iv) By Proposition 3.1.1.(*iv*), one has $p + q \sim_0 p \oplus q$, and therefore $[p + q]_0 = [p \oplus q]_0 = [p]_0 + [q]_0$ by (*i*).

v) If $[p]_0 = [q]_0$, then by Proposition 3.2.1.(*iv*) there exists $r \in \mathcal{P}_\infty(\mathcal{C})$ such that $[p]_{\mathcal{D}} + [r]_{\mathcal{D}} = [q]_{\mathcal{D}} + [r]_{\mathcal{D}}$. Hence $[p \oplus r]_{\mathcal{D}} = [q \oplus r]_{\mathcal{D}}$, and then $p \oplus r \sim_0 q \oplus r$. It thus follows that $p \sim_s q$.

Conversely, if $p \sim_s q$, then there exists $r \in \mathcal{P}_\infty(\mathcal{C})$ such that $p \oplus r \sim_0 q \oplus r$. By (*i*) one gets that $[p]_0 + [r]_0 = [q]_0 + [r]_0$, and because $K_0(\mathcal{C})$ is a group we conclude that $[p]_0 = [q]_0$. □

Proposition 3.2.5 (Universal property of K_0)**.** *Let $\mathcal{C}$ be a unital C^*-algebra, and let H be an Abelian group. Suppose that there exists $\nu : \mathcal{P}_\infty(\mathcal{C}) \to H$ satisfying the three conditions:*

(i) $\nu(p \oplus q) = \nu(p) + \nu(q)$ *for any* $p, q \in \mathcal{P}_\infty(\mathcal{C})$,

(ii) $\nu(0_{\mathcal{C}}) = 0$,

(iii) *If* $p, q \in \mathcal{P}_n(\mathcal{C})$ *for some* $n \in \mathbb{N}^*$ *and if* $p \sim_h q \in \mathcal{P}_n(\mathcal{C})$, *then* $\nu(p) = \nu(q)$.

Then there exists a unique group homomorphism $\alpha : K_0(\mathcal{C}) \to H$ such that the diagram

$$\begin{array}{ccc} \mathcal{P}_\infty(\mathcal{C}) & & \\ {\scriptstyle [\cdot]_0}\downarrow & \searrow^{\nu} & \\ K_0(\mathcal{C}) & \xrightarrow[\alpha]{} & H \end{array}$$

is commutative.

The proof of this statement is provided the proof of [RLL00, Prop. 3.1.8] to which we refer.

3.3 Functoriality of K_0

Let us now consider two unital C^*-algebras $\mathcal{C}$ and $\mathcal{Q}$, and let $\varphi : \mathcal{C} \to \mathcal{Q}$ be a $*$-homomorphism. As already seen in Section 1.3, φ extends to a $*$-homomorphism $\varphi : M_n(\mathcal{C}) \to M_n(\mathcal{Q})$ for any $n \in \mathbb{N}^*$. Again, the same notation is used for the original morphism and for its extensions. Since $*$-homomorphisms map projections to projections, one infers that φ maps $\mathcal{P}_\infty(\mathcal{C})$ into $\mathcal{P}_\infty(\mathcal{Q})$. Let us then define the map $\nu : \mathcal{P}_\infty(\mathcal{C}) \to K_0(\mathcal{Q})$ by $\nu(p) := [\varphi(p)]_0$ for any $p \in \mathcal{P}_\infty(\mathcal{C})$. Since ν satisfies the three conditions of Proposition 3.2.5 with $H = K_0(\mathcal{Q})$ one infers that there exists a unique group homomorphism $K_0(\varphi) : K_0(\mathcal{C}) \to K_0(\mathcal{Q})$ given by

$$K_0(\varphi)([p]_0) = [\varphi(p)]_0 \tag{3.3}$$

for any $p \in \mathcal{P}_\infty(\mathcal{C})$. In other words, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{P}_\infty(\mathcal{C}) & \xrightarrow{\varphi} & \mathcal{P}_\infty(\mathcal{Q}) \\ \Big\downarrow{\scriptstyle [\cdot]_0} & & \Big\downarrow{\scriptstyle [\cdot]_0} \\ K_0(\mathcal{C}) & \xrightarrow[K_0(\varphi)]{} & K_0(\mathcal{Q}). \end{array}$$

With this construction at hand, we can now state and prove the main result on functoriality. Here, the functor K_0 associates with any unital C^*-algebra $\mathcal{C}$ the Abelian group $K_0(\mathcal{C})$. For two unital C^*-algebras $\mathcal{C}$ and $\mathcal{Q}$ one sets $0_{\mathcal{C}\to\mathcal{Q}}$ for the map sending all elements of $\mathcal{C}$ to $0 \in \mathcal{Q}$, and $0_{K_0(\mathcal{C})\to K_0(\mathcal{Q})}$ for the map sending all elements of $K_0(\mathcal{C})$ to the identity element in $K_0(\mathcal{Q})$.

Proposition 3.3.1 (Functoriality of K_0 (unital case)). *Let $\mathcal{J}$, $\mathcal{C}$ and $\mathcal{Q}$ be unital C^*-algebras. Then*

(i) $K_0(\mathrm{id}_\mathcal{C}) = \mathrm{id}_{K_0(\mathcal{C})}$,

(ii) If $\varphi : \mathcal{J} \to \mathcal{C}$ and $\psi : \mathcal{C} \to \mathcal{Q}$ are $*$-homomorphisms, then

$$K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi),$$

(iii) $K_0(\{0\}) = \{0\}$,

(iv) $K_0(0_{\mathcal{C}\to\mathcal{Q}}) = 0_{K_0(\mathcal{C})\to K_0(\mathcal{Q})}$.

Proof. By using (3.3) one can check that for any $p \in \mathcal{P}_\infty(\mathcal{C})$ and any $q \in \mathcal{P}_\infty(\mathcal{J})$ the equalities

$$K_0(\mathrm{id}_\mathcal{C})([p]_0) = [p]_0, \qquad K_0(\psi \circ \varphi)([q]_0) = \big(K_0(\psi) \circ K_0(\varphi)\big)([q]_0)$$

hold. Then, by taking the standard picture of K_0 (equality (3.2)) into account, one readily deduces the statement (*i*) and (*ii*).

iii) One has $\mathcal{P}_n(\{0\}) = \{0_n\}$, with 0_n the zero (and single) element of $M_n(\{0\})$. Since the zero projections $0 = 0_1, 0_2, \dots$ are all $\sim_0$-equivalent, it follows that $\mathcal{D}(\{0\}) = \{[0]_\mathcal{D}\}$. As a consequence, one deduces that $K_0(\{0\}) = \mathcal{G}\big(\{[0]_\mathcal{D}\}\big) = \{0\}$.

iv) Since $0_{\mathcal{C}\to\mathcal{Q}} = 0_{0\to\mathcal{Q}} \circ 0_{\mathcal{C}\to 0} : \mathcal{C} \to \{0\} \to \mathcal{Q}$, the statement (*iv*) can be deduced from (*ii*) and (*iii*). □

For two C^*-algebras $\mathcal{C}$ and $\mathcal{Q}$, two $*$-homomorphisms $\varphi_0 : \mathcal{C} \to \mathcal{Q}$ and $\varphi_1 : \mathcal{C} \to \mathcal{Q}$ are said to be *homotopic*, written $\varphi_0 \sim_h \varphi_1$, if there exists a path of $*$-homomorphisms $t \mapsto \varphi(t)$ with $\varphi(0) = \varphi_0$ and $\varphi(1) = \varphi_1$ such that for any $a \in \mathcal{C}$ the map $[0,1] \ni t \mapsto [\varphi(t)](a) \in \mathcal{Q}$ is continuous. In this case, one also says that $t \mapsto \varphi(t)$ is *pointwise continuous*. The two C^*-algebras $\mathcal{C}$ and $\mathcal{Q}$ are said to be *homotopy equivalent* if there exist two $*$-homomorphisms $\varphi : \mathcal{C} \to \mathcal{Q}$ and $\psi : \mathcal{Q} \to \mathcal{C}$ such that $\psi \circ \varphi \sim_h \mathrm{id}_\mathcal{C}$ and $\varphi \circ \psi \sim_h \mathrm{id}_\mathcal{Q}$. In this case one says that

$$\mathcal{C} \xrightarrow{\varphi} \mathcal{Q} \xrightarrow{\psi} \mathcal{C} \tag{3.4}$$

is a *homotopy* between $\mathcal{C}$ and $\mathcal{Q}$.

Proposition 3.3.2 (Homotopy invariance of K_0 (unital case))**.** *Let $\mathcal{C}$ and $\mathcal{Q}$ be unital C^*-algebras.*

(i) If $\varphi, \psi : \mathcal{C} \to \mathcal{Q}$ are homotopic $*$-homomorphisms, then $K_0(\varphi) = K_0(\psi)$,

(ii) If $\mathcal{C}$ and $\mathcal{Q}$ are homotopy equivalent, then $K_0(\mathcal{C})$ is isomorphic to $K_0(\mathcal{Q})$. More specifically, if (3.4) *is a homotopy between $\mathcal{C}$ and $\mathcal{Q}$, then $K_0(\varphi) : K_0(\mathcal{C}) \to K_0(\mathcal{Q})$ and $K_0(\psi) : K_0(\mathcal{Q}) \to K_0(\mathcal{C})$ are isomorphisms, with $K_0(\varphi)^{-1} = K_0(\psi)$.*

Exercise 3.3.3. *Provide a proof of Proposition 3.3.2, with the possible help of [RLL00, Prop. 3.2.6].*

Our next aim is to show that K_0 preserves exactness of the short exact sequence obtained by adjoining a unit to a unital C^*-algebra. This result will be useful when defining the K_0-group for a non-unital C^*-algebra.

For two C^*-algebras $\mathcal{C}$ and $\mathcal{Q}$, two $*$-homomorphisms $\varphi : \mathcal{C} \to \mathcal{Q}$ and $\psi : \mathcal{C} \to \mathcal{Q}$ are said to be *orthogonal to each other* or *mutually orthogonal*, written $\varphi \perp \psi$, if $\varphi(a)\psi(b) = 0$ for any $a, b \in \mathcal{C}$.

Lemma 3.3.4. *If $\mathcal{C}$ and $\mathcal{Q}$ are unital C^*-algebras, and if $\varphi : \mathcal{C} \to \mathcal{Q}$ and $\psi : \mathcal{C} \to \mathcal{Q}$ are mutually orthogonal $*$-homomorphisms, then $\varphi + \psi : \mathcal{C} \to \mathcal{Q}$ is a $*$-homomorphism, and $K_0(\varphi + \psi) = K_0(\varphi) + K_0(\psi)$.*

Proof. One readily check that $\varphi + \psi : \mathcal{C} \to \mathcal{Q}$ is a $*$-homomorphism. In addition, the $*$-homomorphism $\varphi : M_n(\mathcal{C}) \to M_n(\mathcal{Q})$ and $\psi : M_n(\mathcal{C}) \to M_n(\mathcal{Q})$ are also orthogonal, for any $n \in \mathbb{N}^*$. By using then Proposition 3.2.4.(*iv*) we obtain for any $p \in \mathcal{P}_n(\mathcal{C})$:

$$\begin{aligned} K_0(\varphi + \psi)([p]_0) &= [(\varphi + \psi)(p)]_0 = [\varphi(p) + \psi(p)]_0 \\ &= [\varphi(p)]_0 + [\psi(p)]_0 = K_0(\varphi)([p]_0) + K_0(\psi)([p]_0). \end{aligned}$$

This shows that $K_0(\varphi+\psi) = K_0(\varphi) + K_0(\psi)$. □

Lemma 3.3.5. *For any unital C^*-algebra $\mathcal{C}$, the split exact sequence*

$$0 \longrightarrow \mathcal{C} \overset{\iota}{\hookrightarrow} \widetilde{\mathcal{C}} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0$$

induces a split exact sequence

$$0 \longrightarrow K_0(\mathcal{C}) \xrightarrow{K_0(\iota)} K_0(\widetilde{\mathcal{C}}) \underset{K_0(\lambda)}{\overset{K_0(\pi)}{\rightleftarrows}} K_0(\mathbb{C}) \longrightarrow 0 \tag{3.5}$$

Proof. Recall from the proof of Lemma 2.2.4 that if $\tilde{\mathbf{1}}$ denotes the unit of $\widetilde{\mathcal{C}}$ and if $\mathbf{1}$ denotes the unit of $\mathcal{C}$, then $1 := \tilde{\mathbf{1}} - \mathbf{1}$ is a projection in $\widetilde{\mathcal{C}}$. In addition, $\widetilde{\mathcal{C}} = \mathcal{C} + \mathbb{C}1$, with $a1 = 1a = 0$ for any $a \in \mathcal{C}$. Let us then define the $*$-homomorphisms $\mu : \widetilde{\mathcal{C}} \to \mathcal{C}$ and $\lambda' : \mathbb{C} \to \widetilde{\mathcal{C}}$ by $\mu(a + \alpha 1) := a$ and $\lambda'(\alpha) := \alpha 1$ for any $a \in \mathcal{C}$ and $\alpha \in \mathbb{C}$. One readily infers that

$$\mathrm{id}_{\mathcal{C}} = \mu \circ \iota, \quad \mathrm{id}_{\widetilde{\mathcal{C}}} = \iota \circ \mu + \lambda' \circ \pi, \quad \pi \circ \iota = 0_{\mathcal{C}\to\mathbb{C}}, \quad \pi \circ \lambda = \mathrm{id}_{\mathbb{C}},$$

and the $*$-homomorphisms $\iota \circ \mu$ and $\lambda' \circ \pi$ are orthogonal to each other. Proposition 3.3.1 and Lemma 3.3.4 then lead to

$$\begin{aligned}
0_{K_0(\mathcal{C})\to K_0(\mathbb{C})} &= K_0(0_{\mathcal{C}\to\mathbb{C}}) = K_0(\pi) \circ K_0(\iota), && (3.6)\\
\mathrm{id}_{K_0(\mathbb{C})} &= K_0(\mathrm{id}_{\mathbb{C}}) = K_0(\pi \circ \lambda) = K_0(\pi) \circ K_0(\lambda), && (3.7)\\
\mathrm{id}_{K_0(\mathcal{C})} &= K_0(\mathrm{id}_{\mathcal{C}}) = K_0(\mu \circ \iota) = K_0(\mu) \circ K_0(\iota), && (3.8)\\
\mathrm{id}_{K_0(\widetilde{\mathcal{C}})} &= K_0(\mathrm{id}_{\widetilde{\mathcal{C}}}) = K_0(\iota \circ \mu + \lambda' \circ \pi)\\
&= K_0(\iota) \circ K_0(\mu) + K_0(\lambda') \circ K_0(\pi). && (3.9)
\end{aligned}$$

Now, the split exactness of (3.5) follows from these equalities. Indeed, the injectivity of $K_0(\iota)$ follows from (3.8). If $g \in \mathsf{Ker}\big(K_0(\pi)\big)$, one infers from (3.9) that $g = K_0(\iota)\big(K_0(\mu)(g)\big)$, which shows that g belongs to $\mathsf{Ran}\,\big(K_0(\iota)\big)$. Since by (3.6) one also gets $\mathsf{Ran}\,\big(K_0(\iota)\big) \subset \mathsf{Ker}\,\big(K_0(\pi)\big)$, these two inclusions mean that $\mathsf{Ran}\,\big(K_0(\iota)\big) = \mathsf{Ker}\,\big(K_0(\pi)\big)$. Finally, the surjectivity of $K_0(\pi)$ is a by-product of (3.7), from which one also infers the splitness. □

3.4 Examples

In this section, we introduce the examples discussed in [RLL00, Sec. 3.3] and refer to this book for the proofs.

Consider first a C^*-algebra $\mathcal{C}$ endowed with *a bounded trace* τ, *i.e.* $\tau : \mathcal{C} \to \mathbb{C}$ is a bounded linear map satisfying the *trace property*

$$\tau(ab) = \tau(ba), \qquad \forall a, b \in \mathcal{C}.$$

This trace property implies in particular that $\tau(p) = \tau(q)$ whenever p, q are Murray-von Neumann equivalent projections in $\mathcal{C}$. This trace is also called *positive* if $\tau(a) \geq 0$ whenever $a \in \mathcal{C}^+$. If $\mathcal{C}$ is unital and if $\tau(\mathbf{1}_{\mathcal{C}}) = 1$, then τ is called a *tracial state*.

For any trace τ on a C^*-algebra $\mathcal{C}$, one defines a trace on $M_n(\mathcal{C})$ by setting

$$\tau\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = \sum_{j=1}^{n} \tau(a_{jj}).$$

It thus endows $\mathcal{P}_\infty(\mathcal{C})$ with a map $\tau : \mathcal{P}_\infty(\mathcal{C}) \to \mathbb{C}$, and this map satisfies the three conditions of Proposition 3.2.5. For the last one, recall that the homotopy equivalence implies the Murray-von Neumann equivalence, see Lemma 2.2.9. As a consequence, one infers that there exists a unique group homomorphism $K_0(\tau) : K_0(\mathcal{C}) \to \mathbb{C}$ satisfying for any $p \in \mathcal{P}_\infty(\mathcal{C})$

$$K_0(\tau)([p]_0) = \tau(p). \tag{3.10}$$

Note that if τ is positive, then the r.h.s. of (3.10) is a positive real number, and $K_0(\tau)$ maps $K_0(\mathcal{C})$ into $\mathbb{R}$.

Example 3.4.1. *For any $n \in \mathbb{N}^*$, one has*

$$K_0\big(M_n(\mathbb{C})\big) \cong \mathbb{Z}. \tag{3.11}$$

In fact, if tr *denotes the usual trace already introduced in Exercise 3.1.4, then*

$$K_0(\mathrm{tr}) : K_0\big(M_n(\mathbb{C})\big) \to \mathbb{Z} \tag{3.12}$$

is an isomorphism.

Example 3.4.2. *If $\mathcal{H}$ is an infinite dimensional separable Hilbert space, then we have*

$$K_0\big(\mathcal{B}(\mathcal{H})\big) = \{0\}.$$

Note that this fact is related to the content of Exercise 3.1.5.

Example 3.4.3. *If Ω is a compact, connected and Hausdorff space, then there exists a surjective group homomorphism*

$$\dim : K_0\big(C(\Omega)\big) \to \mathbb{Z} \tag{3.13}$$

which satisfies for $p \in \mathcal{P}_\infty\big(C(\Omega)\big)$ and $x \in \Omega$

$$\dim([p]_0) = \mathrm{tr}\big(p(x)\big).$$

Note that by continuity this number is independent of x. Note also that if Ω is contractible[1] *then the map* (3.13) *is an isomorphism.*

Exercise 3.4.4. *Provide the proofs for the statements of Examples 3.4.1, 3.4.2 and 3.4.3.*

Extension 3.4.5. *Study the K-theory for topological spaces, as presented for example in [RLL00, Sec. 3.3.7].*

[1]The space Ω is contractible if there exists $x_0 \in \Omega$ and a continuous map $\alpha : [0,1] \times \Omega \to \Omega$ such that $\alpha(1,x) = x$ and $\alpha(0,x) = x_0$ for any $x \in \Omega$.