Chapter 9

Complex numbers

9.1 Basic introduction

The aim of this chapter is to provide a very short introduction to complex numbers. One use of complex numbers is to find solutions of the equations $x^2 = -1$, or more generally to find solutions of the equation $ax^2 + bx + c = 0$ for arbitrary $a, b, c \in \mathbb{R}$.

The first step in the construction is based on an analogy with $\mathbb{R}^2$. Note that for simplicity we shall denote the elements of $\mathbb{R}^2$ by (x, y) instead of ${}^t(x, y)$. Let us consider $\mathbb{R}^2$ endowed with the usual addition: $(x, y) + (x', y') = (x + x', y + y')$ for any (x, y) and (x', y') in $\mathbb{R}^2$. We now define a complex multiplication $*$ for these two elements:

$$(x, y) * (x', y') = (xx' - yy', xy' + yx') \quad \in \mathbb{R}^2 \tag{9.1.1}$$

Let us stress that up to now, we had not defined any product of elements of $\mathbb{R}^2$: the scalar product is also taking two elements of $\mathbb{R}^2$ but the result of the scalar product is an element of $\mathbb{R}$, not of $\mathbb{R}^2$!

Since (9.1.1) is rather complicated to remember, let us introduce a symbol i with the only rule that

$$i\,i = i^2 = -1. \tag{9.1.2}$$

We also rewrite (x, y) as $x + iy$. Then, one can again multiply $x + iy$ and $x' + iy'$ by using the common rule of multiplication. One gets

$$\begin{aligned}(x + iy)(x' + iy') &= xx' + (iy)x' + x(iy') + (iy)(iy') \\ &= xx' + i^2 yy' + ixy' + iyx' \\ &= (xx' - yy') + i(xy' + yx'). \end{aligned} \tag{9.1.3}$$

Note that by comparing (9.1.1) with (9.1.3), one observes that the same result is obtained, but (9.1.3) is certainly easier to remember since only usual multiplications are involved. The key point in the construction is the equality mentioned in (9.1.2). Let us mention that the notation z^2 is also used for $z\,z$ (the product of z by itself), and that with this notation, the usual addition can be rewritten as

$$(x + iy) + (x' + iy') = (x + x') + i(y + y'). \tag{9.1.4}$$

We are now ready for introducing the set of complex numbers:

Definition 9.1.1. *One defines*

$$\mathbb{C} := \{z = x + iy \mid x, y \in \mathbb{R}\}$$

endowed with the addition recalled in (9.1.4) *and with the multiplication introduced in* (9.1.3). *This set is called* the set of complex numbers.

Let us stress that we write indifferently $x + iy$ or $x + yi$.

Example 9.1.2. $1+1i$, $7-2i$, $-3+i$, 3, $2i$ *are elements of* $\mathbb{C}$, *where we have identified* 3 *with* $3 + 0i$, $2i$ *with* $0 + 2i$ *and* $-3 + i$ *with* $-3 + 1i$.

By taking into account the identification of x with $x + 0i$, it is clear that $\mathbb{R}$ is included in $\mathbb{C}$. It corresponds to the elements on the horizontal axis in the mentioned analogy of $\mathbb{C}$ with $\mathbb{R}^2$.

Let us still add some examples of multiplications or additions:

Examples 9.1.3. *(i)* $(3 + 2i) + (1 + 1i) = 4 + 3i$,

(ii) $(3 + 2i) + (1 - 3i) = 4 - 1i$,

(iii) $(2 + 2i)(1 + 3i) = 2 + 2i + 6i - 6 = -4 + 8i$,

(iv) $(1 + 2i)(-3 - 2i) = -3 - 6i - 2i + 4 = 1 - 8i$.

Let us now prove an important result about complex numbers. We recall that the notion of a field has been introduced in Definition 3.1.1.

Theorem 9.1.4. $\mathbb{C}$ *is a field.*

Proof. The proof consists in checking the various properties mentioned in Definition 3.1.1. For that purpose, let us set $z = x + iy$ and $z_j = x_j + iy_j$ for $j \in \{1, 2, 3\}$ and with $x, y, x_j, y_j \in \mathbb{R}$. Then one has

(i) $z_1 + z_2 \in \mathbb{C}$ and $z_1 z_2 \in \mathbb{C}$, which means that these operations are internal,

(ii) $(z_1+z_2)+z_3 = z_1+(z_2+z_3)$ and $(z_1 z_2)z_3 = z_1(z_2 z_3)$, as shown in Exercise 9.1. This corresponds to the associativity of the addition and of the complex multiplication

(iii) $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$, as shown in Exercise 9.2. This means that the addition and the complex multiplication are commutative,

(iv) Let us set $0 \equiv 0 + 0i$ and $1 \equiv 1 + 0i$, which correspond to the usual 0 and 1 of $\mathbb{R}$. Then it is easily observed that $z + 0 = z$ and that $1\, z = z$. This property corresponds to the existence of identity elements for the addition and for the complex multiplication,

(v) Observe that if $z = x + iy \in \mathbb{C}$, then $-x - iy$ also belongs to $\mathbb{C}$ and one has $(x+iy)+(-x-iy) = 0$. Thus $-x-iy$ is the inverse of $x+iy$ for the addition. For the inverse of $x+iy$ with respect to the addition, let us assume that $x+iy \neq 0$, which means that $(x,y) \neq (0,0)$, and let us consider the complex number $\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$ $\mathbb{C}$. This element is well defined since its denominator is different from 0. Then one observes that

$$(x+iy)\Big(\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\Big) = \frac{x^2+y^2}{x^2+y^2} + i\frac{-xy+xy}{x^2+y^2} = 1$$

Thus one has $(x+iy)^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$, when the inverse with respect to the complex multiplication is considered.

(vi) The distributivity of complex multiplication with respect to the addition of complex numbers is shown in Exercise 9.1.

□

In addition to $\mathbb{R}$ we have thus a second field at our disposal, the field $\mathbb{C}$ of complex numbers. The corresponding complex vector spaces and linear maps on complex vector spaces are briefly studied in Section 7.4. Let us still emphasize one formula which has been derived in the previous proof: for any $z = x+iy \in \mathbb{C}$ with $z \neq 0$ one has

$$(x+iy)^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}. \tag{9.1.5}$$

Let us now introduce some notations. For any $z = x+iy \in \mathbb{C}$, one sets $\Re(z) := x$ and $\Im(z) := y$ for the *real part* and the *imaginary part* of z. We also introduce the *complex conjugate* $\overline{z}$ of z by

$$\overline{z} = \overline{x+iy} := x - iy.$$

Note that in the mentioned analogy of $\mathbb{C}$ with $\mathbb{R}^2$, it corresponds to taking the image of z by a symmetry along the horizontal axis. Then, with this concept of complex conjugate, it is easily observed that

$$\Re(z) = \frac{z+\overline{z}}{2} \qquad \text{and} \qquad \Im(z) = \frac{z-\overline{z}}{2}.$$

For any complex number $z = x+iy$ we also define $|z| := \sqrt{x^2+y^2}$ and set

$$z = r\big(\cos(\theta) + i\sin(\theta)\big)$$

with $r = |z|$, $x = r\cos(\theta)$ and $y = r\sin(\theta)$. This is called *the polar coordinate representation* of the complex number z. The number $r \equiv |z|$ is called the *norm* or the *modulus* of z, and θ its *argument*, *i.e.* $\theta = \arg(z)$. We also introduce the notation

$$e^z = e^{x+iy} := e^x\big(\cos(y) + i\sin(y)\big).$$

These notations will be used in the Exercises, and they are very useful tools for complex numbers.

Remark 9.1.5. *Let us emphasize that $\mathbb{C}$ has no ordering. Indeed, even if $\mathbb{R}$ has a ordering (one says for example that $-2 < 4$), it is impossible to compare two complex numbers as for example $3 - 2i$ and $4 + i$.*

Let us now provide one of the basic result for complex numbers, which is part of the motivation for introducing them.

Proposition 9.1.6. *For any $a + ib \in \mathbb{C}$, there exists $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq z_2$ (except if $a + ib = 0$) such that $z_1^2 = z_2^2 = a + ib$. In other words, every complex number has two distinct square roots.*

Proof. Let us first observe that for any $a, b \in \mathbb{R}$, one has

$$a + \sqrt{a^2 + b^2} \geq 0 \qquad \text{and} \qquad -a + \sqrt{a^2 + b^2} \geq 0.$$

Thus, one can define $x := \sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}$ with the usual square root of positive numbers, and also $y := \sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}$ with the usual square root. We then set

$$z_1 := x + i\mu y \qquad \text{and} \qquad z_2 := -x - i\mu y$$

with $\mu = 1$ if $b \geq 0$ and $\mu = -1$ if $b < 0$. It only remains to check with the definition of the complex multiplication that $z_1^2 = a + ib$ and that $z_2^2 = a + ib$ as well. □

By using the well-known formula for the solutions of a second degree equation, one infers that:

Corollary 9.1.7. *The equation $az^2 + bz + c = 0$ has always two solutions in $\mathbb{C}$.*

Let us finally mention that this corollary is at the root of the fundamental theorem of algebra asserting that any polynomial of degree n has n solutions in $\mathbb{C}$.

9.2 Exercises

Exercise 9.1. *Let z_1, z_2, z_3 be three complex numbers. Show that*

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

and that

$$(z_1 z_2)z_3 = z_1(z_2 z_3).$$

These properties correspond to the associativity *of the addition and of the complex multiplication. In addition, check that $z_1(z_2+z_3) = z_1 z_2 + z_1 z_3$. This property corresponds to* the distributivity *of the complex multiplication with respect to the addition of complex numbers.*

Exercise 9.2. *For $z_1, z_2 \in \mathbb{C}$, show that $z_1 + z_2 = z_2 + z_1$ and that $z_1\, z_2 = z_2\, z_1$. These properties correspond to the commutativity of the addition and of the complex multiplication.*

Exercise 9.3. *Compute the real part and the imaginary part of the number $\frac{3+2i}{2-3i}$. Same question with the number $\frac{1}{i} + \frac{3}{1+i}$ and the number $\sqrt{1+i}$.*

Exercise 9.4. *Find all solutions of the equation $z^4 = -1$.*

Exercise 9.5. *For any $z_1, z_2 \in \mathcal{C}$, show that $|z_1 z_2| = |z_1||z_2|$ and that*

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2).$$

Exercise 9.6. *Deduce from the previous exercise* de Moivre's formula*: for any $n \in \mathbb{N}$ and for $z = r\big(\cos(\theta) + i\sin(\theta)\big)$ one has*

$$z^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big).$$

Exercise 9.7. *Deduce that for any complex number $z = r\big(\cos(\theta) + i\sin(\theta)\big)$, the n-th roots of z are given by*

$$z_j := \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi j}{n}\right) + i\sin\left(\frac{\theta + 2\pi j}{n}\right)\right]$$

for $j \in \{0, 1, \dots, n-1\}$.

Exercise 9.8. *Show the following properties:*

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$,
2. $\overline{z_1 z_2} = \overline{z_1}\;\overline{z_2}$,
3. $z\,\overline{z} = |z|^2$,
4. $z^{-1} = \overline{z}/|z|^2$ *whenever* $z \neq 0$,

5. *$\Re(z) = (z + \overline{z})/2$ and $\Im(z) = (z - \overline{z})/(2i)$, where $\Re(z)$ and $\Im(z)$ are the real and the imaginary part of z.*

Exercise 9.9. *Show also that $|\overline{z}| = |z|$ and that $\arg(\overline{z}) = -\arg(z)$.*

Exercise 9.10. *Show the following properties:*

1. *$e^{z_1+z_2} = e^{z_1} e^{z_2}$ for any $z_1, z_2 \in \mathcal{C}$,*

2. *e^z is never equal to 0,*

3. *$|e^{x+iy}| = e^x$,*

4. *$e^{i\pi} = -1$ (Euler's identity, and "one of the most beautiful formula in mathematics").*