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Exercise 1.3.3 For the operator A_n defined in (1.3.1), give an upper estimate for $\|A_n\|$ and compute A_n^*.

Solution

One has

$$\|A_n\| = \sup_{\substack{f,g\in\mathcal{H} \\ \|f\|=\|g\|=1}} |\langle A_n f, g\rangle| \leq \sup_{\substack{f,g\in\mathcal{H} \\ \|f\|=\|g\|=1}} \sum_{j=1}^{n} \|f\|\|g_j\|\|g\|\|h_j\| \leq \sum_{j=1}^{n} \|g_j\|\|h_j\|.$$

For the adjoint, one observes that

$$\langle A_n^* f, g\rangle = \langle f, A_n g\rangle = \langle f, \sum_{k=1}^{n} \langle g, g_k\rangle h_k\rangle = \overline{\langle \sum_{k=1}^{n} \langle g, g_k\rangle h_k\, , f\rangle} = \sum_{k=1}^{n} \overline{\langle g, g_k\rangle\langle h_k, f\rangle}$$

$$= \sum_{k=1}^{n} \langle f, h_k\rangle\langle g_k, g\rangle = \langle \sum_{k=1}^{n} \langle f, h_k\rangle g_k\, , g\rangle \Rightarrow A_n^* f = \sum_{k=1}^{n} \langle f, h_k\rangle g_k.$$

Exercise 1.6.5 For any closed linear operator A and for any $z_1, z_2 \in \rho(A)$, show the first resolvent equation, namely

$$(A - z_1)^{-1} - (A - z_2)^{-1} = (z_1 - z_2)(A - z_1)^{-1}(A - z_2)^{-1}.$$

Proof

One easily observes that

$$(A - z_1)^{-1}(A - z_2)^{-1} = \big((A - z_2)(A - z_1)\big)^{-1} = \big((A - z_1)(A - z_2)\big)^{-1}$$

$$= (A - z_2)^{-1}(A - z_1)^{-1} \Rightarrow \begin{cases} (A - z_1)^{-1} = (A - z_2)(A - z_1)^{-1}(A - z_2)^{-1} \\ (A - z_2)^{-1} = (A - z_1)(A - z_1)^{-1}(A - z_2)^{-1} \end{cases}$$

$$\Rightarrow (A - z_1)^{-1} - (A - z_2)^{-1} = (z_1 - z_2)(A - z_1)^{-1}(A - z_2)^{-1}.$$

Exercise 2.1.6 By using the Neumann series, show that $Inv(\mathcal{B})$ is an open set in a unital Banach algebra $\mathcal{B}$, and that the map $Inv(\mathcal{B}) \ni A \mapsto A^{-1} \in \mathcal{B}$ is differentiable.

Proof

Let $\forall A \in Inv(\mathcal{B})$ and $\|\Delta A\| < \|A^{-1}\|^{-1}$, then $A_1 = A + \Delta A$ is invertible, and ${A_1}^{-1} = (1 + A^{-1}\Delta A)^{-1}A^{-1}$, since

$$A_1 = A + \Delta A = A(1 + A^{-1}\Delta A) \text{ and } \|-A^{-1}\Delta A\| \leq \|A^{-1}\|\|\Delta A\| < \|A^{-1}\|^{-1}\|A^{-1}\| = 1.$$

Therefore, $Inv(\mathcal{B})$ is an open set.

$$(A + \Delta A)^{-1} - A^{-1} = (1 + A^{-1}\Delta A)^{-1}A^{-1} - A^{-1} = \big(1 - (1 - A^{-1}\Delta A + (A^{-1}\Delta A)^2 - \cdots)\big)A^{-1}$$

$$= (A^{-1}\Delta A - (A^{-1}\Delta A)^2 + \cdots)A^{-1} = A^{-1}\Delta A A^{-1} + o(\Delta A)$$

Therefore, the map $Inv(\mathcal{B}) \ni A \mapsto A^{-1} \in \mathcal{B}$ is differentiable.

Exercise 3.1.12 We state in this exercise a couple of useful formulas which can be deduced from the definition of the modular function. Let $f \in C_c(G)$ and $x \in G$:

$$\int_G f(xy)d\mu(y) = \int_G f(y)d\mu(y),$$

$$\int_G f(yx)d\mu(y) = \Delta(x)^{-1}\int_G f(y)d\mu(y),$$

$$\int_G \Delta(y^{-1})f(y^{-1})d\mu(y) = \int_G f(y)d\mu(y).$$

Proof

Let $f = f^+ - f^-, f \in C_c(G)$, where $f^+(x) = \begin{cases} f(x), f(x) \geq 0 \\ 0, f(x) < 0 \end{cases}, f^-(x) = \begin{cases} 0, f(x) > 0 \\ -f(x), f(x) \leq 0 \end{cases} \Rightarrow$ $f^+, f^- \in C_c^+(G)$

$$\int_G f^+(xy)d\mu(y) = \int_G [L_{x^{-1}}f^+](y)d\mu(y) = \int_G f^+(y)d\mu(y)$$

$$\int_G f^+(yx)d\mu(y) = \int_G [R_x f^+](y)d\mu(y) = \Delta(x^{-1})\int_G f^+(y)d\mu(y) = \Delta(x)^{-1}\int_G f^+(y)d\mu(y)$$

by $\int_G R_y f d\mu = \Delta(y^{-1})\int_G f d\mu$.

Similarly, we can prove that this is true for f^-, so it is true for f.

$$J(g) \coloneqq \int_G \Delta(z^{-1})g(z^{-1})d\mu(z), (\lambda_s g)(t) \coloneqq g(s^{-1}t), g \in C_C(G)$$

$$J(\lambda_s g) = \int_G \Delta(z^{-1})g(s^{-1}z^{-1})d\mu(z) = \int_G \Delta(s)\Delta(s^{-1})\Delta(z^{-1})g(s^{-1}z^{-1})d\mu(z)$$

$$= \Delta(s)\int_G \Delta((zs)^{-1})g((zs)^{-1})d\mu(z) = \Delta(s)\int_G \Delta((zs)^{-1})g((zs)^{-1})d\mu(z)$$

$$= \int_G \Delta(z^{-1})g(z^{-1})d\mu(z)$$

As a result, we know that $J(g)$ is left-invariant. Therefore, $\exists c \in \mathbb{R}_+$ such that

$$J(g) = c\int_G g(z)d\mu(z).$$

Let $g \in C_C(G), g \geq 0, h(z) \coloneqq g(z) + \Delta(z^{-1})g(z^{-1}) \Rightarrow h(z) \in C_C(G), h(z) = \Delta(z^{-1})h(z^{-1}) \Rightarrow$ $\int_G h(z)d\mu(z) = J(h) \Rightarrow c = 1.$

Exercise 4.1.2 For any $\varphi \in C_b(\mathbb{R}^d)$, show that the spectrum of the multiplication operator $\varphi(X)$ coincides with the closure of $\varphi(\mathbb{R}^d)$ in $\mathbb{C}$.

Proof

$\lambda \in \varphi(\mathbb{R}^d) \Rightarrow \exists x \in \mathbb{R}^d\ s.t. \big(\lambda - \varphi(X)\big)f(x) = \big(\lambda - \varphi(x)\big)f(x) = 0, \forall f \in \mathcal{D}\left(\varphi(\mathbb{R}^d)\right) \Rightarrow \big(\lambda - \varphi(X)\big)$ is not invertible.

$$\Rightarrow \lambda \in \sigma\big(\varphi(X)\big)$$

$$\Rightarrow \sigma\big(\varphi(X)\big) = \overline{\sigma\big(\varphi(X)\big)} \supset \overline{\varphi(\mathbb{R}^d)}$$

$$\lambda \notin \overline{\varphi(\mathbb{R}^d)} \Rightarrow \big(\lambda 1(X) - \varphi(X)\big)^{-1} = g(X), g(x) = \frac{1}{\lambda - \varphi(x)} \Rightarrow \lambda \notin \sigma\big(\varphi(X)\big)$$

$$\Rightarrow \overline{\varphi(\mathbb{R}^d)} \supset \sigma\big(\varphi(X)\big)$$

$$\Rightarrow \sigma\big(\varphi(X)\big) = \overline{\varphi(\mathbb{R}^d)}.$$

Exercise 5.1.5 Check carefully the statements contained in the previous example.

Proof

$$\begin{aligned}(\tilde{\pi}(\varphi_1\varphi_2)h)(x) &= \pi\big(\theta_x(\varphi_1\varphi_2)\big)h(x) = \pi\big(\theta_x(\varphi_1)\theta_x(\varphi_2)\big)h(x) = \pi\big(\theta_x(\varphi_1)\big)\pi\big(\theta_x(\varphi_2)\big)h(x)\\ &= (\tilde{\pi}(\varphi_1)\tilde{\pi}(\varphi_2)h)(x)\end{aligned}$$

$$\begin{aligned}\big(\widetilde{U}_x\widetilde{U}_y h\big)(z) &= \pi\big(\omega(z,x)\big)\big(\widetilde{U}_y h\big)(z+x) = \pi\big(\omega(z,x)\big)\pi\big(\omega(z+x,y)\big)h(z+y+x)\\ &= \pi\big(\omega(z+x,y)\omega(z,x)\big)h(z+y+x)\\ &= \pi\Big(\theta_z\big(\omega(x,y)\big)\omega(z,x+y)\Big)h(z+y+x)\\ &= \pi\Big(\theta_z\big(\omega(x,y)\big)\Big)\pi\big(\omega(z,x+y)\big)h(z+y+x) = \big(\tilde{\pi}\big(\omega(y,x)\big)\widetilde{U}_{x+y}h\big)(z).\end{aligned}$$

$$\begin{aligned}\langle\widetilde{U}_y h, g\rangle &= \int_G \langle\pi\big(\omega(x,y)\big)h(x+y), g(x)\rangle dx = \int_G \langle h(x+y), \Big(\pi\big(\omega(x,y)\big)\Big)^* g(x)\rangle\, dx\\ &= \int_G \langle h(x), \Big(\pi\big(\omega(x-y,y)\big)\Big)^{-1} g(x-y)\rangle\, dx\\ &= \int_G \langle h(x), \pi\Big(\big(\omega(x-y,y)\big)^{-1}\Big) g(x-y)\rangle\, dx\\ &\Rightarrow \big(\widetilde{U}_y{}^* h\big)(x) = \pi\Big(\big(\omega(x-y,y)\big)^{-1}\Big)h(x-y).\end{aligned}$$

$$\begin{aligned}\big(\widetilde{U}_y\tilde{\pi}(\varphi)\widetilde{U}_y{}^* h\big)(x) &= \pi\Big(\big(\omega(x,y)\big)\Big)\big(\tilde{\pi}(\varphi)\widetilde{U}_y{}^* h\big)(x+y)\\ &= \pi\Big(\big(\omega(x,y)\big)\Big)\pi\Big(\theta_{x+y}(\varphi)\Big)\big(\widetilde{U}_y{}^* h\big)(x+y)\\ &= \pi\Big(\big(\omega(x,y)\big)\Big)\pi\Big(\theta_{x+y}(\varphi)\Big)\pi\Big(\big(\omega(x,y)\big)^{-1}\Big)h(x) = \pi\Big(\theta_{x+y}(\varphi)\Big)h(x)\end{aligned}$$

$$\Big(\tilde{\pi}\Big(\theta_y(\varphi)\Big)h\Big)(x) = \pi\Big(\theta_x\Big(\theta_y(\varphi)\Big)\Big)h(x) = \pi\Big(\theta_{x+y}(\varphi)\Big)h(x).$$

Exercise 6.1.1 Check that if $f(x,\xi) = f(\xi)$, then $\mathfrak{D}_{\mathrm{p}}(f) = f(D)$, while if $f(x,\xi) = f(x)$, then $\mathfrak{D}_{\mathrm{p}}(f) = f(X)$.

Proof

$$\begin{aligned}f(x,\xi) = f(\xi) \Rightarrow \big(\mathfrak{D}_{\mathrm{p}}(f)u\big)(x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}}^d} e^{i(x-y)\eta} f(\eta)\, u(y)dyd\eta\\ &= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{ix\eta}\left(\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\widehat{\mathbb{R}}^d} e^{-ix\eta}\, u(y)dy\right) f(\eta)d\eta\\ &= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{ix\eta} f(\eta)\hat{u}(\eta)d\eta = F^*(f(X)Fu)(x) = (f(D)u)(x).\end{aligned}$$

Define

$$g_y(x) := f\left(\frac{x+y}{2}\right).$$

Since $f(x,\xi) = f(x)$, we can get

$$
\begin{aligned}
\big(\mathfrak{D}_{\mathrm{p}}(f)u\big)(x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}}^d} e^{i(x-y)\eta} f\left(\frac{x+y}{2}\right)u(y)dyd\eta \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\eta}\int_{\widehat{\mathbb{R}}^d} e^{-iy\eta} f\left(\frac{x+y}{2}\right)u(y)dyd\eta = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{ix\eta}\widehat{g_x u}(\eta)d\eta \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{ix\eta}(\widehat{g_x} * \hat{u})(\eta)d\eta = F^*(\widehat{g_x} * \hat{u})(x) = g_x(x)u(x) = f(x)u(x) \\
&= (f(X)u)(x).
\end{aligned}
$$

Exercise 7.1.3 Work out the details of the previous proof, and in particular show that ω^B satisfies the two conditions (5.1.1) and (5.1.2).

Proof

Recall that

$$\theta_x\big(\omega^B(y,z)\big)(q) = \omega^B(y,z)(q+x) = \omega^B(q+x;y,z).$$

We show that

$$\omega^B(q;x+y,z)\omega(q;x,y) = \omega^B(q+x;y,z)\omega^B(q;x,y+z).$$

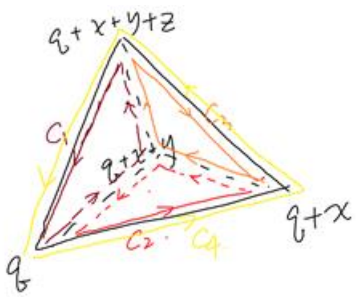
It is sufficient to show that

$$
\begin{aligned}
&\Gamma^B(\langle q,q+x+y,q+x+y+z\rangle) + \Gamma^B(\langle q,q+x,q+x+y\rangle) \\
&\qquad = \Gamma^B(\langle q+x,q+x+y,q+x+y+z\rangle) + \Gamma^B(\langle q,q+x,q+x+y+z\rangle).
\end{aligned}
$$

In fact

$$
\begin{aligned}
&\Gamma^B(\langle q,q+x+y,q+x+y+z\rangle) + \Gamma^B(\langle q,q+x,q+x+y\rangle) \\
&\qquad - \Gamma^B(\langle q+x,q+x+y,q+x+y+z\rangle) - \Gamma^B(\langle q,q+x,q+x+y+z\rangle) = \\
&\qquad = \int_{C^1+C^2-C^3-C^4} B = \int_T dB = 0.
\end{aligned}
$$

The last equation is by the Stokes' Theorem, where T is the interior of the tetrahedron.



$$\omega^B(q;x,0) = 1.$$

It is enough to show that

$$\Gamma^B(\langle q,q+x,q+x\rangle) = 0.$$

In fact

$$\Gamma^B(\langle q,q+x,q+x\rangle) = \int_l B = 0.$$

It is because B is a closed continuous 2-form but l is a segment from q to $q+x$.

Similarly, we can show

$$\omega^B(q;0,x) = 1.$$