C^*-ALGEBRAS, GROUP ACTIONS AND CROSSED PRODUCTS (LECTURE NOTES)

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Contents

Date: February 22, 2007.

1. Introduction

These notes are a draft of a course offered to students of third, fourth and fifth year of theoretical physics. The aim of the course is to familiarize students with concepts of abstract theory of C*-algebras and group representations. It is not our objective to give a full fledged course on operator algebras. In fact, we hope that interested students will either take it upon themselves to read some of the standard textbooks or choose to attend other courses where the fundamentals such as Gelfand's theory of commutative Banach algebras, functional calculus or elements of the theory of von Neumann algebras would be presented.

We optimalized this course towards minimal prerequisites. Our potential student should have attended a mathematical analysis course where the notion of a Banach space had been introduced and an algebra course where the concept of a group had been discussed (even if only finite groups were mentioned). It would be very helpful if a course on mathematical methods in physics where topics like representation theory of finite and compact groups as well as the elementary theory of Lie groups were presented had been taken by our prospective audience. Nevertheless, this last blessing is not absolutely necessary to understand our subject. An open mind and desire to seek information on individual basis will, on the other hand, be indispensable.

The main focus of our presentation is the notion of a crossed product of a C*-algebra by an action of a locally compact group. The subject is presented in the language of abstract theory of C*-algebras with emphasis on the universal property of the crossed product. We will introduce and thoroughly analyze the notion of the multiplier algebra of a C*-algebra and that of a morphism between C*-algebras. Then we will study some simple aspects of the theory of actions of groups on C*-algebras and representations of groups in C*-algebras. Many examples tieing these subjects to the theory of unitary representations of groups will be given.

After that our central object — the crossed product — will be constructed and a number of well known examples will be given. We hope that these examples will show how easily very interesting C*-algebras can be obtained as crossed products.

The following sections can be divided into two groups. The first group consists of sections in which practically all statements are given with complete proofs. It is worth noting that these proofs are usually different form those given in literature. This group includes Sections 4, 5 and 6 (except for subsection 6.6).

The remaining sections are included for merely informative purpose or to provide the bare facts we simply cannot do without. In section 2 we gather the few definitions and pieces of notation we are using throughout the notes. Section 3 lists some elements of the theory of topological groups and, specifically, locally compact groups that are necessary in the study of crossed products. This includes a presentation of the notion of the Haar integral and its properties. In that section we only give proofs of some simple general results about topological groups, since these facts are not usually included in texts on operator algebras. There are no proofs of statements concerning Haar integrals as the author feels that G.K. Pedersen's books [3, 4] are much better sources on that topic than any of his own efforts.

The reader is urged to glance at Section 2 and then to start reading from Section 4 and continue through Sections 5, 6 returning to Section 3 only when necessary. Section 7 is intended only to encourage the reader's interest in the advanced theory and results of the wonderfully rich world of group actions on C*-algebras.

The reference list for these notes is drastically short, but we do feel that it is fairly complete.

2. Preliminaries

In these notes we shall deal with many topological spaces. Let us list here the standing assumptions and introduce some notation.

Let X be a topological space. The space of all continuous complex valued functions on X will be denoted by $\mathrm{C}(X)$. The subspace of $\mathrm{C}(X)$ consisting of bounded continuous functions on X will be written as $\mathrm{C_b}(X)$. A very often used symbol $\mathrm{C}_\infty(X)$ will denote the space consisting of all those continuous functions $X \to \mathbb{C}$ satisfying

$$\forall\, \varepsilon > 0 \text{ the set } \bigl\{x \in X \bigm| |f(x)| \geq \varepsilon\bigr\} \text{ is compact.}$$

Finally the symbol $\mathrm{C_c}(X)$ will denote the space of continuous functions on X with compact supports. We have the obvious chain of inclusions

$$\mathrm{C_c}(X) \subset \mathrm{C}_\infty(X) \subset \mathrm{C_b}(X) \subset \mathrm{C}(X) \tag{2.1}$$

For a general topological space X some of the spaces (2.1) can be trivial. However if X is a locally compact space all of the spaces (2.1) are relevant for the study of X. Here and throughout these notes we are including the Hausdorff separation axiom in the definition of local compactness (and compactness). Note that in case of a compact space X all the spaces (2.1) are the same.

3. Topological groups

3.1. **Groups with topology.** Let G be a group. We say that G is a *topological group* if there is a topology on the set G such that the map

$$G \times G \ni (t,s) \longmapsto t^{-1}s \in G$$

is continuous. It is a standard fact that the topology of a topological group is *translationally invariant,* i.e. the maps $s \mapsto ts$ and $s \mapsto st$ are homeomorphisms for all $t \in G$. In particular, this means that the topology is uniquely determined by the, so called, *local base* or more precisely the local base of neighborhoods of $e \in G$ (e will denote the neutral element of a group throughout these notes). Such a local base can be chosen to consists of sets $\mathcal{U}$ such that $\mathcal{U}^{-1} = \mathcal{U}$. Such sets are called *symmetric.* This fact follows from the next lemma.

Lemma 3.1. *Let G be a toplogical group. Then for any neighborhood $\mathcal{W}$ of e there exists a symmetric neighborhood $\mathcal{U}$ of e such that $\mathcal{U}^2 \subset \mathcal{W}$.*

Proof. Since $e^2 = e$ and group multiplication is continuous there are neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of e such that $\mathcal{U}_1\mathcal{U}_2 \subset \mathcal{W}$. Take

$$\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2 \cap \bigl(\mathcal{U}_1^{-1}\bigr) \cap \bigl(\mathcal{U}_2^{-1}\bigr)$$

and use the fact that the inverse operation is continuous. □

The most important class of groups we shall deal with will be the *locally compact groups.* These are the topological groups whose topology is locally compact.

3.2. **Uniform continuity.**

Definition 3.2. Let G be a topological group.

(1) A function $f : G \to \mathbb{C}$ is *right uniformly continuous* if for any $\varepsilon > 0$ there exists a neighborhood $\mathcal{U}$ of $e \in G$ such that $s^{-1}t \in \mathcal{U}$ implies $\bigl|f(s) - f(t)\bigr| < \varepsilon$.
(2) A function $f : G \to \mathbb{C}$ is *left uniformly continuous* if $t \mapsto f\bigl(t^{-1}\bigr)$ is right uniformly continuous.
(3) A function $f : G \to \mathbb{C}$ is *uniformly continuous* if it is both left and right uniformly continuous.

Proposition 3.3. *Let G be a locally compact topological group and let $f \in \mathrm{C}_\infty(G)$. Then f is uniformly continuous. In particular any compactly supported function is uniformly continuous.*

Proof. Let us first show that f is right uniformly continuous i.e. we want to show that for a fixed $\varepsilon > 0$ there is a neighborhood $\mathcal{U}$ of $e \in G$ such that $s^{-1}t \in \mathcal{U}$ implies $\bigl|f(s) - f(t)\bigr| < \varepsilon$.

For any $a \in G$ there exists a neighborhood $\mathcal{W}_a$ of e such that $s \in a\mathcal{W}_a$ implies $\bigl|f(s) - f(a)\bigr| < \frac{\varepsilon}{2}$ (this is true for any $f \in \mathrm{C}(G)$). Let $\mathcal{U}_a$ be a neighborhood of e such that $\mathcal{U}_a^2 \subset \mathcal{W}_a$ (see Lemma 3.1).

Let $K \subset G$ be a compact set such that $|f| < \frac{\varepsilon}{2}$ outside K. Then

$$\Bigl(k\mathcal{U}_k\Bigr)_{k \in K}$$

is an open covering of K. Therefore there is a finite set $A\subset K$ such that

$$K\subset\bigcup_{a\in A}a\mathcal{U}_a.$$

Let

$$\mathcal{U}=\bigcap_{a\in A}\mathcal{U}_a.$$

Now take $s\in K$, $t\in G$ such that $s^{-1}t\in\mathcal{U}$ and choose $a\in A$ such that $s\in a\mathcal{U}_a$. Then $\bigl|f(s)-f(a)\bigr|<\frac{\varepsilon}{2}$, and since $a^{-1}t=(a^{-1}s)(s^{-1}t)\in\mathcal{U}_a\mathcal{U}\subset\mathcal{W}_a$, we also have $\bigl|f(t)-f(a)\bigr|<\frac{\varepsilon}{2}$. We have thus shown that for $s\in K$, $t\in G$ the fact that $s^{-1}t\in\mathcal{U}$ implies that

$$\bigl|f(s)-f(t)\bigr|<\varepsilon. \tag{3.1}$$

If $s,t\in G$ with $s^{-1}t\in\mathcal{U}$ then either one of them is in K or both are outside K. In the former case we can assume that $s\in K$ (we can always rename s and t) and we have (3.1). In the latter case $\bigl|f(s)\bigr|,\bigl|f(t)\bigr|<\frac{\varepsilon}{2}$, so we also have (3.1).

We have thus shown that f is right uniformly continuous. Since the function $t\mapsto f(t^{-1})$ also belongs to $\mathrm{C}_\infty(G)$ we see that f is left uniformly continuous as well. □

We shall need the concept of uniform continuity in the proof of Lemma 5.3 (1) and of Theorem 5.9 in Subsection 5.2. There we will be speaking about Banach space valued uniformly continuous functions (the generalization from $\mathbb{C}$-valued functions is straightforward). We will then use the fact that left uniform continuity of a function f is equivalent to the fact that the functions

$$s\longmapsto f(us)$$

converge to f uniformly as $u\to e$ in G. Therefore, for $f\in\mathrm{C}_\infty(G)$, given $\varepsilon>0$, there is a neighborhood $\mathcal{V}$ of $e\in G$ such that for any $s\in G$ and $t\in\mathcal{V}$ we have $\bigl|f(s)-f(t^{-1}s)\bigr|<\varepsilon$.

3.3. **Integration.** The crucial development in the theory of locally compact groups was the proof of existence of an *left invariant measure,* so called *Haar measure,* on any locally compact group. Just as crucial is the fact that such a measure is unique up to rescaling.

We will not develop this theory here, nor are we going to prove any of the statements about Haar measure. The spirit of there notes is such that we merely want to state the results which we will use in very special cases (e.g. we will only integrate continuous functions with compact support).

Let G be a locally compact group. Then there exists a Radon integral on G (positive linear functional on $\mathrm{C_c}(G)$), denoted by $\int_G\cdot\,dh^L$, such that

$$\int_G f(s)\,dh^L(s)=\int_G f(ts)\,dh^L(s)$$

for any $t\in G$. The measure h^L on the σ-algebra of Borel subsets of G is called a *Haar measure.*

- All left Haar measures on G are proportional,
- the Haar measure of any compact subset of G is finite.
- the Haar measure of any nonempty open subset of G is strictly positive (although the zero measure is invariant, we don't call it the Haar measure),
- since G^{opp} is a locally compact group, there exists a right Haar measure h^R such that

  $$\int_G f(s)\,dh^R(s)=\int_G f(st)\,dh^R(s)$$

  for all $t\in G$, a right Haar measure is also unique up to rescaling,
- it is customary to normalize the Haar measure on a compact group G so that $h^L(G)=1$; for discrete G we normalize so that $h^L(\{e\})=1$.

It is obvious that on an Abelian group the left Haar measure is also a right one. The same is true for all compact and all discrete groups. In those cases we will write h instead of h^L (or h^R).

Let us fix a locally compact group G with a left Haar measure h^L. For any $t \in G$ the map

$$\mathrm{C_c}(G) \ni f \longmapsto \int\limits_G f(st)\,dh^L(s)$$

is a left invariant Radon integral, an by the uniqueness of h^L must correspond to a multiple of h^L. The factor of proportionality is denoted by $\Delta(t)$:

$$\int\limits_G f(st)\,dh^L(s) = \Delta(t) \int\limits_G f(s)\,dh^L(s).$$

This way we obtain a function Δ on G with values in the strictly positive real numbers. This function is called the *modular function* of G. Note that the definition of Δ does not depend on the choice of the Haar measure on G.

Proposition 3.4. *Let G be a locally compact group and let Δ be the modular function of G. Then*

(1) *Δ is a continuous homomorphism from G into the multiplicative group of strictly positive real numbers,*
(2) *for any $f \in \mathrm{C_c}(G)$ we have*
$$\int\limits_G f(t^{-1})\Delta(t)^{-1}\,dh^L(t) = \int\limits_G f(t)\,dh^L(t),$$
(3) *the map*
$$\mathrm{C_c}(G) \ni f \longmapsto \int\limits_G f(s)\Delta(s)^{-1}\,dh^L(s)$$
is a right invariant Radon integral on G.

The proofs of all the statements above can be found in [4] (except for the existence of the Haar measure for which we refer to [2]).

The existence of Haar measure brings analysis into the world of locally compact groups. The new subject obtained in this way is referred to as *abstract harmonic analysis.* One of the elementary notions of harmonic analysis is that of convolution of functions. If G is a locally compact group and h^L is a left Haar measure on G then to any two integrable functions f and g we can associate a new function

$$(f \star g)(t) = \int\limits_G f(s)g(s^{-1}t)\,dh^L(s).$$

It turns out that $f \star g$ is integrable and we even have the following result.

Lemma 3.5. *Let G be a locally compact group and let $f, g \in L^1(G)$. Then*

$$\|f \star g\|_1 \leq \|f\|_1 \|g\|_1.$$

The function $f \star g$ is called the *convolution* of the functions f and g. What is even more important than the fact that $f \star g \in L^1(G)$ is that convolution is an associative product on $L^1(G)$ which makes it into a *Banach algebra* (see Subsection 4.1).

3.4. **Examples.** The most important examples of locally compact groups are the ones which do not require the abstract theory described above. The first of these is the group $\mathbb{R}$ of all real numbers with addition as group operation. On it we consider the metric topology given by the absolute value. The left (and right) Haar integral is given by the ordinary Lebesgue integral.

The second very important example is the group $\mathbb{T}$ of complex numbers of absolute value 1 with multiplication as the group operation and the topology inherited from $\mathbb{C}$. The left (and right) Haar integral is

$$\int\limits_{\mathbb{T}} f(s)\,dh(s) = \frac{1}{2\pi} \int\limits_0^{2\pi} f\big(e^{2\pi \mathrm{i} t}\big)\,dt.$$

A related example is the discrete additive group $\mathbb{Z}$ of the integers. The left (and right) Haar measure in this example is the counting measure.

All the above groups are Abelian. A non Abelian example is the group G of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$. This is the group of all invertible affine transformations of $\mathbb{R}$ into itself. The topology on G is inherited from $M_2(\mathbb{R}) \approx \mathbb{R}^4$. This group is not Abelian and it has different right and left Haar integrals. We have

$$\int_G f(s)\,dh^L(s) = \int_G f\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) |a|^{-2} da\,db,$$

$$\int_G f(s)\,dh^R(s) = \int_G f\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) |a|^{-1} da\,db$$

up to rescaling (integrals on the right hand side a simply Lebesgue integrals over $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}$). The modular function is

$$\Delta\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) = |a|.$$

The last example opens up the whole world of Lie groups. It is not difficult to construct a left invariant volume form on any Lie group which will quickly lead to a left Haar integral. Thus, for example, for $G = GL(2, \mathbb{R})$ we have

$$\int_G f(s)\,dh^L(s) = \int_G f\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) \frac{d\alpha\,d\beta\,d\gamma\,d\delta}{(\alpha\delta - \beta\gamma)^2}$$

(here, in fact, the integral is both left and right invariant).

4. Some C*-algebra theory

All vector spaces considered here will be over the field of complex numbers.

4.1. C*-algebras.

Definition 4.1.

(1) An *algebra* (over $\mathbb{C}$) is a vector space A endowed with a product $A \times A \ni (a, b) \mapsto ab \in A$ such that
  - $a(bc) = (ab)c$ for all $a, b, c \in A$ (associativity),
  - $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in A$ (distributivity),
  - $(\alpha a)(\beta b) = (\alpha\beta)(ab)$ for all $\alpha, \beta \in \mathbb{C}$ and $a, b \in A$ (compatibility with scalar multiplication).

(2) A *normed algebra* is an algebra A with a norm $A \ni a \mapsto \|a\|$ such that $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$.

(3) A $*$-*algebra* is an algebra A provided with a map $* : A \ni a \mapsto a^* \in A$ such that
  - $(a + b)^* = a^* + b^*$ for all $a, b \in A$,
  - $(\alpha a)^* = \overline{\alpha} a^*$ for all $\alpha \in \mathbb{C}$ and $a \in A$,
  - $(ab)^* = b^* a^*$ for all $a, b \in A$,
  - $(a^*)^* = a$ for all $a \in A$.

  The map $a \mapsto a^*$ is called the *involution*.

(4) A *normed* $*$-*algebra* is a normed algebra A which is at the same time a $*$-algebra and the norm satisfies $\|a^*\| = \|a\|$ for all $a \in A$.

(5) A *Banach algebra* is a normed algebra which is complete in its norm.

(6) A *Banach* $*$-*algebra* is a normed $*$-algebra which is complete in its norm.

(7) A C*-*algebra* is a Banach $*$-algebra A such that
$$\|a^*a\| = \|a\|^2 \tag{4.1}$$
for all $a \in A$.

The identity (4.1) is called the C*-*identity*.

Lemma 4.2. *Let A be a* C*-*algebra. Then for any $a \in A$ we have*
$$\|a\| = \sup_{\substack{b\in A\\ \|b\|=1}} \|b^*a\|.$$

Proof. For $b \in A$ with $\|b\| = 1$ we have $\|b^*a\| \leq \|a\|$. On the other hand for $b = \frac{1}{\|a\|}a^*$ we obtain $\|b^*a\| = \|a\|$. □

The next result is a standard part of any course on C*-algebras. Its proof uses continuous functional calculus for self adjoint elements of C*-algebras. We will omit the proof, but the result will be used in what follows.

Proposition 4.3. *Let A be a* C*-*algebra and let $\mathcal{J}$ be a left ideal of A. There exists a net $(e_\lambda)_{\lambda\in\Lambda}$ of elements of $\mathcal{J}$ such that $\|e_\lambda\| \leq 1$ for all $\lambda \in \Lambda$ and*
$$\|x - xe_\lambda\| \xrightarrow[\lambda\in\Lambda]{} 0 \tag{4.2}$$
for any $x \in \mathcal{J}$. The net $(e_\lambda)_{\lambda\in\Lambda}$ can be chosen to consist of positive elements and to be increasing.

Taking in Proposition 4.3 $\mathcal{J} = A$ we obtain a net $(e_\lambda)_{\lambda\in\Lambda}$ of elements of A such that (4.2) holds for any $x \in A$. Such a net is called an approximate identity for A. If A is unital then $(e_\lambda)_{\lambda\in\Lambda}$ converges in norm to the unit of A (enough to put $x = I$ in (4.2)).

4.1.1. *Examples.*

Example 4.4. Let X be a locally compact space (a locally compact space will, by definition, be a Hausdorff space). The space $\mathrm{C}_\infty(X)$ consisting of all those continuous functions $X \to \mathbb{C}$ satisfying
$$\forall\, \varepsilon > 0 \text{ the set } \bigl\{x \in X \bigm| |f(x)| \geq \varepsilon\bigr\} \text{ is compact}$$
is an algebra over $\mathbb{C}$ (cf. Section 2). With the norm
$$\|f\| = \sup_{x\in X}\bigl|f(x)\bigr|$$
it is a Banach algebra. Moreover $\mathrm{C}_\infty(X)$ is a Banach $*$-algebra with involution $f \mapsto f^*$ defined by
$$f^*(x) = \overline{f(x)}$$
for all $x \in X$. It is straightforward to verify that $\mathrm{C}_\infty(X)$ is in fact a C*-algebra.

We have the following facts

- $\mathrm{C}_\infty(X)$ has a unit if and only if X is a compact space.
- Let $(K_\lambda)_{\lambda\in\Lambda}$ be a family of compact subsets of X directed by inclusion and such that $\bigcup\limits_{\lambda\in\Lambda} = X$. For each λ there exists a function $f_\lambda \in \mathrm{C}_\infty(X)$ such that for each $\lambda \in \Lambda$ we have $\|f_\lambda\| \leq 1$ and $f_\lambda(y) = 1$ for any $y \in \mathcal{K}_\lambda$. The net $(f_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for $\mathrm{C}_\infty(X)$.
- If A is a C*-algebra and the multiplication in A is commutative then there exists a unique locally compact space X such that A is isometrically isomorphic to $\mathrm{C}_\infty(X)$. More precisely there exists a unique locally compact space X and a linear isomorphism $\Phi : \mathrm{C}_\infty(X) \to A$ such that $\Phi(f_1f_2) = \Phi(f_1)\Phi(f_2)$ and $\Phi(f^*) = \Phi(f)^*$ for all $f, f_1, f_2 \in \mathrm{C}_\infty(X)$. The map Φ is an isometry of the Banach space $\mathrm{C}_\infty(X)$ onto A. This result is referred to as the *first Gelfand-Naimark theorem.*

Example 4.5. Let H be a Hilbert space. The Banach space $\mathrm{B}(H)$ is a unital algebra over $\mathbb{C}$. The multiplication is given by composition of operators. The hermitian conjugation of operators endows $\mathrm{B}(H)$ with structure of a $*$-algebra. It is immediate that $\mathrm{B}(H)$ is a Banach $*$-algebra. Let us prove that $\mathrm{B}(H)$ is a C*-algebra.

Recall that for any vector $\xi \in H$ we have $\|\xi\| = \sup\limits_{\|\eta\|=1} \bigl|(\eta|\xi)\bigr|$. Now it is clear that for any $a \in \mathrm{B}(H)$ we have $\|a^*a\| \leq \|a^*\|\|a\| = \|a\|^2$. On the other hand

$$\begin{aligned}\|a\|^2 &= \bigl(\sup_{\|\xi\|=1} \|a\xi\|\bigr)^2\\ &= \sup_{\|\xi\|=1} \|a\xi\|^2\\ &= \sup_{\|\xi\|=1} (a\xi|a\xi)\\ &= \sup_{\|\xi\|=1} (\xi|a^*a|\xi)\\ &\leq \sup_{\|\xi\|=1}\sup_{\|\eta\|=1} \bigl|(\eta|a^*a|\xi)\bigr|\\ &= \sup_{\|\xi\|=1} \|a^*a\xi\| = \|a^*a\|.\end{aligned}$$

We have the following facts:

- Any $\|\cdot\|$-closed $*$-subalgebra of $\mathrm{B}(H)$ is a C*-algebra.
- A particularly important C*-algebra contained in $\mathrm{B}(H)$ is the algebra $\mathcal{K}(H)$ consisting of all compact operators (an operator $c \in \mathrm{B}(H)$ is *compact* if it is a limit of a sequence of operators with finite dimensional ranges).
- Let $(e_\imath)_{\imath\in\mathcal{I}}$ be an orthonormal basis of H. Let Λ be the set of finite subsets of $\mathcal{I}$ ordered by inclusion. Then Λ is a directed set. For $\lambda \in \Lambda$ let p_λ be the orthogonal projection onto the subspace spanned by elements of λ. Then for each λ the operator $p_\lambda \in \mathcal{K}(H)$ and $(p_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for $\mathcal{K}(H)$.
- For any C*-algebra A there exists a Hilbert space H and a linear isomorphism Φ from A onto a $\|\cdot\|$-closed $*$-subalgebra of $\mathrm{B}(H)$. This isomorphism is multiplicative, $*$-preserving and isometric. In other words any C*-algebra can be realized as a C*-algebra of operators on a Hilbert space. This result is referred to as the *second Gelfand-Naimark theorem.*

4.2. **Multipliers.**

Definition 4.6. Let A be a C*-algebra. A linear operator $m : A \to A$ is called a *multiplier* of A if for any $a \in A$ there exists a $c \in A$ such th for all $b \in A$ we have

$$a^*mb = c^*b. \tag{4.3}$$

The set of all multipliers of A is denoted by $\mathrm{M}(A)$.

Proposition 4.7. *Let A be a C*-algebra.*

1. $\mathrm{M}(A)$ *is a unital algebra.*
2. *For each $m \in \mathrm{M}(A)$ and for any $a \in A$ the element $c \in A$ such that* (4.3) *holds for all $b \in A$ is unique. Denoting this unique c by m^*a we define an operator $m^* : A \to A$. Then $m^* \in \mathrm{M}(A)$ and the map $m \mapsto m^*$ endows $\mathrm{M}(A)$ with a $*$-algebra structure.*
3. *Each multiplier is a bounded operator on A.*
4. *The operator norm turns $\mathrm{M}(A)$ into a unital C*-algebra.*
5. *For any $a \in A$ the operator $L_a : A \ni b \mapsto ab \in A$ belongs to $\mathrm{M}(A)$ and the left regular representation $L : A \ni a \mapsto L_a \in \mathrm{M}(A)$ is an injective $*$-homomorphism onto a closed essential ideal.*

Proof. It is straightforward to check that a product and sum of multipliers is a multiplier. The identity map on A is a multiplier. This takes care of (1).

Ad (2). Assume that we have $c, d \in A$ such that

$$c^*b = d^*b$$

for all $b \in A$. Then $(c-d)^*b = 0$ for all b. In particular for $b = (c-d)$ we have $(c-d)^*(c-d) = 0$. Thereofore

$$\|c-d\|^2 = \big\|(c-d)^*(c-d)\big\| = \|0\| = 0,$$

so $c = d$. Thus given $a \in A$ the element c such that (4.3) holds for all $b \in A$ is unique.

Let m^* be the map sending a to the unique c satisfying (4.3). Then m^* is a linear operator on A and for $a, b \in A$ we have

$$(a^*m^*b)^* = (m^*b)^*a = b^*ma,$$

so taking $*$ of both sides of this equation we see that $m^* \in \mathrm{M}(A)$ and $(m^*)^* = m$. It is easy to check that $*$ is an involution on $\mathrm{M}(A)$.

Ad (3). To see that $\mathrm{M}(A) \subset \mathrm{B}(A)$ it is enough to show that each $m \in \mathrm{M}(A)$ is a closed map. Assume that (a_n) is a sequence in A and $a_n \xrightarrow[n\to\infty]{} a$. Assume also that $ma_n \xrightarrow[n\to\infty]{} b$. Then for any $c \in A$

$$c^*ma_n = (m^*c)^*a_n \xrightarrow[n\to\infty]{} (m^*c)^*a = c^*ma,$$

but the left hand side converges to c^*b. It follows, as in the reasoning proving Statement (2), that $ma = b$. It is also easy to see that the involution $m \to m^*$ is an isometry for the operator norm.

Ad (4). We must show that the C*-identity is satisfied and that $\mathrm{M}(A)$ is complete. Take $m \in \mathrm{M}(A)$ and $c \in A$ with $\|a\| = 1$. We have by lemma 4.2

$$\begin{aligned}\|mc^*\|^2 &= \big\|(mc)^*mc\big\| \\ &= \|c^*m^*mc\| \\ &\le \sup_{\substack{b\in B\\ \|b\|=1}} \|b^*m^*mc\| = \|m^*mc\|\end{aligned}$$

Taking sup over c with $\|c\| = 1$ we obtain $\|m\|^2 \le \|m^*m\|$. On the other hand we of course have $\|m^*m\| \le \|m\|^2$.

Now let us check completeness of $\mathrm{M}(A)$. Let $(m_n)_{n\in\mathbb{N}}$ be a sequence of multipliers of A which is Cauchy for the operator norm. Then there exists its limit $t \in \mathrm{B}(A)$. Since $*$ is isometric, there also exists a limit s of $(m_n^*)_{n\in\mathbb{N}}$. Let us see that $t \in \mathrm{M}(A)$. Take $a, b \in A$.

$$a^*tb = \lim_{n\to\infty} a^*m_nb = \lim_{n\to\infty} (m_n^*a)^*b = (sa)^*b.$$

It follows that $t \in \mathrm{M}(A)$ and $t^* = s$.

Ad (5). One can check that for any $a \in A$ the operator L_a of left multiplication by a is a multiplier and that the adjoint of L_a is L_{a^*}. Moreover the map $a \mapsto L_a$ is linear and multiplicative. It is also injective because $L_a = 0$ implies $L_aa^* = 0$, so that $\|a\|^2 = \|a^*\|^2 = \|aa^*\| = \|L_aa^*\| = 0$ and $a = 0$.

Let $m \in \mathrm{M}(A)$. We have for any $a, b \in A$

$$L_amb = amb = (a^*)^*mb = (m^*a^*)^*b = L_{(m^*a^*)^*}b,$$

so L_am belongs to the image L. Similarly

$$mL_ab = mab = L_{ma}b.$$

It follows that $L(A)$ is an ideal in $\mathrm{M}(A)$. Any injective $*$-homomorphism of C*-algebras is isometric, but we can also see directly that $\|L_a\| = \|a\|$ (cf. Lemma 4.2). Consequently $L(A)$ is closed in $\mathrm{M}(A)$.

Let $\mathcal{J}$ be an ideal in $\mathrm{M}(A)$. If $\mathcal{J} \cap L(A) = \{0\}$ then $\mathcal{J} = \{0\}$ because if $j \in \mathcal{J}$ then jL_a belongs to $\mathcal{J} \cap L(A)$, so $jL_a = 0$. On the other hand we have already checked that $jL_a = L_{ja}$. If this is 0 for all $a \in A$ then $ja = 0$ for all a and this means that j is the zero operator on A. □

From now on we shall consider any C*-algebra A as a subset of its multiplier algebra $\mathrm{M}(A)$. It therefore makes sense to consider extending maps defined on A to $\mathrm{M}(A)$. The identity map id_A of A onto itself will be denoted by I_A whenever it is considered as an element of $\mathrm{M}(A)$.

Proposition 4.8. *Let A be a C*-algebra. Consider A as a subset of $\mathrm{M}(A)$. Then $A = \mathrm{M}(A)$ if and only if A is unital.*

Proof. A is an ideal in $\mathrm{M}(A)$. Therefore if A has a unit I then $E = I_A - I$ generates an ideal $\mathcal{J}$ which has zero intersection with A. But A is essential in $\mathrm{M}(A)$, so $\mathcal{J}$ is $\{0\}$. Since $E = E\,I_A \in \mathcal{J}$ we have $E = 0$, i.e. $I = I_A$.

Conversely, if $A = \mathrm{M}(A)$, then clearly A has a unit. □

4.2.1. *Examples.*

Example 4.9. Let X be a locally compact space. The C*-algebra of multipliers of $\mathrm{C}_\infty(X)$ is the algebra $\mathrm{C_b}(X)$ consisting of all bounded continuous functions $X \to \mathbb{C}$. The norm, multiplication and involution of $\mathrm{C_b}(X)$ are given by the same expressions as for elements of $\mathrm{C}_\infty(X)$.

Note that $\mathrm{C}_\infty(X) = \mathrm{M}\big(\mathrm{C}_\infty(X)\big)$ if and only if X is a compact space.

The C*-algebra $\mathrm{C_b}(X)$ is unital and commutative. Therefore, by the first Gelfan-Naimark theorem, there exists a compact space βX such that

$$\mathrm{C_b}(X) = \mathrm{C}(\beta X). \tag{4.4}$$

This space has been known to topologists since the nineteen thirties. The space βX is called the *Stone-Čech compactification* of X. It can be defined as the maximal compactification of X, but formula (4.4) can just as well serve as the definition of βX.

Lemma 4.10. *Let H be a Hilbert space. Let $\xi, \eta \in H$. Then $\big\| |\xi\,)\,(\,\eta| \big\| = \|\xi\|\|\eta\|$.*

Proof.

$$\begin{aligned}\big\| |\xi\,)\,(\,\eta| \big\| &= \sup_{\|\zeta\|=1} \big\| |\xi\,)\,(\,\eta|\;\zeta \big\| \\ &= \sup_{\|\zeta\|=1} \big\| (\eta|\zeta)\,\xi \big\| \\ &= \|\xi\| \sup_{\|\zeta\|=1} \big|(\eta|\zeta)\big| = \|\xi\|\|\eta\|.\end{aligned}$$

□

Example 4.11. Let H be a Hilbert space. The C*-algebra of multipliers of $\mathcal{K}(H)$ coincides with $\mathrm{B}(H)$. Notice first that any bounded operator m on H defines a multiplier $\mathcal{K}(H) \ni x \mapsto mx \in \mathcal{K}(H)$. The adjoint of this multiplier is left multiplication by m^*.

Now let $t \in \mathrm{M}\big(\mathcal{K}(H)\big)$. Take $\xi, \eta, \zeta, \psi \in H$ and let $f^{\zeta,\eta}_{\psi,\xi} = \big(|\zeta\,)\,(\,\psi|\big)t\big(|\xi\,)\,(\,\eta|\big)$. Then $f^{\zeta,\eta}_{\psi,\xi} \in \mathcal{K}(H)$. Moreover $\operatorname{ran} f^{\zeta,\eta}_{\psi,\xi} \subset \mathbb{C}\zeta$. Since

$$|\zeta\,)\,(\,\psi|\;t\;|\xi\,)\,(\,\eta| = \big(|\psi\,)\,(\,\zeta|\big)^* t\;|\xi\,)\,(\,\eta| = \big(t^*\;|\psi\,)\,(\,\zeta|\big)^*\;|\xi\,)\,(\,\eta|\,,$$

we see that $\ker f^{\zeta,\eta}_{\psi,\xi} \supset \{\eta\}^\perp$. It follows that

$$f^{\zeta,\eta}_{\psi,\xi} = \lambda_{\psi,\xi}\,|\zeta\,)\,(\,\eta|\,.$$

Clearly $\lambda_{\psi,\xi}$ is linear in ξ and antilinear in ψ. The computation

$$|\lambda_{\psi,\xi}|\|\xi\|\|\eta\| = |\lambda_{\psi,\xi}|\big\||\xi\,)\,(\,\eta|\big\| = \|f^{\zeta,\eta}_{\psi,\xi}\| \le \big\||\zeta\,)\,(\,\psi|\big\|\|m\|\big\||\xi\,)\,(\,\eta|\big\| = \|\zeta\|\|\psi\|\|m\|\|\xi\|\|\eta\|$$

shows that the sesquilinear form $(\psi, \zeta) \mapsto \lambda_{\psi,\xi}$ is bounded. Therefore there exists a bounded operator t° on H such that

$$(\psi|t^\circ|\xi) = \lambda_{\psi,\xi}$$

for all $\psi, \xi \in H$. Moreover the formula

$$|\zeta\,)\,(\,\psi|\;t\;|\xi\,)\,(\,\eta| = (\psi|t^\circ|\xi)\,|\zeta\,)\,(\,\eta| = |\zeta\,)\,(\psi|t^\circ|\xi)\,(\,\eta|$$

shows that the action of t coincides with left multiplication by t° on a dense set of compact operators.

4.3. Morphisms of C^*-algebras.

Notation 4.12. Let B be a C^*-algebra and let $\mathcal{X},\mathcal{Y}$ be subspaces of B. The symbol $\mathcal{X}\mathcal{Y}$ will denote the linear span of the set

$$\{xy \,|\, x\in\mathcal{X},\ y\in\mathcal{Y}\}$$

Definition 4.13. Let A and B be C^*-algebras. A *morphism* from A to B is a $*$-homomorphism $\Phi: A\to \mathrm{M}(B)$ such that $\Phi(A)B$ is dense in B. The set of all morphisms from A to B is denoted by $\mathrm{Mor}(A,B)$.

Proposition 4.14. *Let A and B be C^*-algebras and let $\Phi\in\mathrm{Mor}(A,B)$. Then there exists a unique extension of Φ to a unital $*$-homomorphism from $\mathrm{M}(A)$ to $\mathrm{M}(B)$. If Φ is injective (as a map from A to $\mathrm{M}(B)$) then so is its extension to $\mathrm{M}(A)$.*

Proof. Since Φ is a morphism, we know that $\Phi(A)B$ is dense in B. For $m\in\mathrm{M}(A)$ we will first define $\Phi(m)$ as a linear operator on $\Phi(A)B$ by

$$\Phi(m)\left(\sum_{k=1}^{N}\Phi(a_k)b_k\right)=\sum_{k=1}^{N}\Phi(ma_k)b_k. \tag{4.5}$$

We must now check that this operator is well defined and that it is bounded.

Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for A. Then

$$\begin{aligned}\left\|\sum_{k=1}^{N}\Phi(ma_k)b_k\right\| &= \lim_{\lambda\in\Lambda}\left\|\sum_{k=1}^{N}\Phi(me_\lambda a_k)b_k\right\|\\ &= \lim_{\lambda\in\Lambda}\left\|\Phi(me_\lambda)\sum_{k=1}^{N}\Phi(a_k)b_k\right\| \\ &\leq \|m\|\left\|\sum_{k=1}^{N}\Phi(a_k)b_k\right\|\end{aligned} \tag{4.6}$$

(we used the fact that any $*$-homomorphism of C^*-algebras is norm decreasing). This computation shows first of all that the definition (4.5) is correct. Indeed, assume that two combinations

$$\sum_{p=1}^{N'}\Phi(a'_p)b'_p \quad\text{and}\quad \sum_{q=1}^{N''}\Phi(a''_q)b''_q \tag{4.7}$$

are equal. Then we write their difference as $\sum\limits_{k=1}^{N}\Phi(a_k)b_k$ and, by (4.6), we get that the result of applying $\Phi(m)$ to both the combinations (4.7) is the same.

Secondly, (4.6) says that $\Phi(m)$ is bounded on $\Phi(A)B$ and consequently it extends to a bounded operator on B.

To see that $\Phi(m)\in\mathrm{M}(B)$ take $a,a'\in A$ and $b,b'\in B$. We have

$$\begin{aligned}\bigl(\Phi(a)b\bigr)^*\Phi(m)\bigl(\Phi(a')b'\bigr) &= \bigl(\Phi(a)b\bigr)^*\Phi(ma')b'\\ &= b^*\Phi(a)^*\Phi(ma')b'\\ &= b^*\Phi(a^*ma')b'\\ &= b^*\Phi\bigl((m^*a)^*a'\bigr)b'\\ &= b^*\Phi(m^*a)^*\Phi\bigl(a'\bigr)b'\\ &= \bigl(\Phi(m^*a)b\bigr)^*\Phi\bigl(a'\bigr)b'.\end{aligned}$$

Using the fact that $\Phi(A)B$ is dense in B we obtain that $\Phi(m)\in\mathrm{M}(B)$ and $\Phi(m)^*=\Phi(m^*)$. It is clear that the extension of Φ we have just defined is a $*$-homomorphism $\mathrm{M}(A)\to\mathrm{M}(B)$. It is equally clear that the unit of $\mathrm{M}(A)$ is mapped to the unit of $\mathrm{M}(B)$.

Uniqueness of extension of Φ is obvious: for any extension $\widetilde{\Phi}$ of Φ to $\mathrm{M}(A)$ and any $m \in \mathrm{M}(A)$, $a \in A$, $b \in B$ we must have

$$\widetilde{\Phi}(m)\Phi(a)b = \widetilde{\Phi}(m)\widetilde{\Phi}(a)b = \widetilde{\Phi}(ma)b = \Phi(ma)b = \Phi(m)\Phi(a)b$$

because $\widetilde{\Phi}$ is a homomorphism coinciding with Φ on $A \subset \mathrm{M}(A)$. The density of $\Phi(A)B$ in B guarantees that $\widetilde{\Phi}$ coincides with the extension defined by (4.5).

Now assume that (the original) Φ is an injective map. The kernel of the extension of Φ to $\mathrm{M}(A)$ is an ideal. But A is an essential ideal in $\mathrm{M}(A)$ and we know that the intersection of A and the kernel of extended Φ is $\{0\}$ (because Φ is injective on A). It follows that the kernel of Φ extended to $\mathrm{M}(A)$ is trivial. □

Existence of canonical extension of a morphism has the following consequences:

- Let A and B be C^*-algebras and let $\Phi \in \mathrm{Mor}(A, B)$. Then for any $m \in \mathrm{M}(A)$ we can consider $\Phi(m)$ which is a multiplier of B.
- If A, B and C are C^*-algebras and $\Phi \in \mathrm{Mor}(A, B)$, $\Psi \in \mathrm{Mor}(B, C)$ then, a priori, Φ is a $*$-homomorphism $A \to \mathrm{M}(B)$ and Ψ is a $*$-homomorphism $B \to \mathrm{M}(C)$. Now we can take the canonical extension of Ψ, i.e. a map $\mathrm{M}(B) \to \mathrm{M}(C)$ and compose it with Φ to obtain a $*$-homomorphism $A \to \mathrm{M}(C)$. It is easy to check that such a composition is a morphism from A to C: since $\overline{\Phi(A)B} = B$ and $\overline{\Psi(B)C} = C$, we have
$$\begin{aligned}\overline{\Psi\big(\Phi(A)\big)C} &= \overline{\Psi\big(\Phi(A)\big)\overline{\Psi(B)C}}\\ &= \overline{\Psi\big(\Phi(A)\big)\Psi(B)C}\\ &= \overline{\Psi\big(\Phi(A)B\big)C}\\ &= \overline{\Psi\big(\overline{\Phi(A)B}\big)C} = \overline{\Psi(B)C} = C.\end{aligned}$$
So described composition is denoted by $\Psi\circ\Phi$. In what follows all compositions of morphisms will be understood precisely in this sense.

A morphism $\Phi \in \mathrm{Mor}(A, B)$ is called an *isomorphism* if there exists $\Psi \in \mathrm{Mor}(B, A)$ such that for any $a \in A$ we have $\Psi\big(\Phi(a)\big) = a$ and for any $b \in B$ we have $\Phi\big(\Psi(b)\big) = b$. Note that in the first formula $\Phi(a)$ is an element of $\mathrm{M}(B)$ and we need to use the conical extension of Ψ to make sense of $\Psi\big(\Phi(a)\big)$. Similarly $\Psi(b) \in \mathrm{M}(A)$ and $\Phi\big(\Psi(b)\big)$ is the result of applying the canonical extension of Φ to $\Psi(b)$.

This definition of an isomorphism is the usual one known from category theory. $\Phi \in \mathrm{Mor}(A, B)$ is an isomorphism if there exists $\Psi \in \mathrm{Mor}(B, A)$ such that $\Psi\circ\Phi = \mathrm{id}_A$ and $\Phi\circ\Psi = \mathrm{id}_B$.

Proposition 4.15. *Let A and B be C^*-algebras and let $\Phi \in \mathrm{Mor}(A, B)$ be an isomorphism. Then the image of the $*$-homomorphism Φ coincides with the subset B of $\mathrm{M}(B)$ and Φ is a $*$-isomorphism of A onto B.*

Proof. Let $\Psi \in \mathrm{Mor}(B, A)$ be the morphism related to Φ as in the definition of an isomorphism of C^*-algebras. Let us denote by $\widetilde{\Phi} : \mathrm{M}(A) \to \mathrm{M}(B)$ and $\widetilde{\Psi} : \mathrm{M}(B) \to \mathrm{M}(A)$ the canonical extensions of Φ and Ψ to multiplier algebras. Our assumptions are that

$$\begin{aligned}\widetilde{\Psi}\big(\Phi(a)\big) &= a,\\ \widetilde{\Phi}\big(\Psi(b)\big) &= b\end{aligned}$$

for all $a \in A$ and $b \in B$. Take $a = \Psi(b_0)a_0$. Then

$$\Phi(a) = \widetilde{\Phi}\big(\Psi(b_0)\big)\Phi(a_0) = b_0\Phi(a_0) \in B.$$

Since elements of the form $\Psi(b_0)a_0$ span a dense subspace of A, we have $\Phi(A) \subset B$. Similarly $\Psi(B) \subset A$.

Now if $a \in A$ then $a = \widetilde{\Psi}\big(\Phi(a)\big)$. We have just shown that $\Phi(a) \in B$, so a is the image of an element of B under $\widetilde{\Psi}$. This of course means that a is really in the image of Ψ and we get $\Psi(B) = A$. Similarly $\Phi(A) = B$.

The fact that Φ is a $*$-isomorphism of A onto B is obvious. □

Example 4.16. Let X and Y be locally compact spaces and let $\Phi_* : X \to Y$ be a continuous map. For any $f \in \mathrm{C}_\infty(Y)$ define the function $\Phi(f)$ on X by the formula

$$\big(\Phi(f)\big)(x) = f\big(\Phi_*(x)\big).$$

One easily checks that $\Phi(f) \in \mathrm{C}_\mathrm{b}(X)$. Moreover the mapping $\Phi : f \mapsto \Phi(f)$ is an element of $\mathrm{Mor}\big(\mathrm{C}_\infty(Y), \mathrm{C}_\infty(X)\big)$.

It can be shown that any $\Psi \in \mathrm{Mor}\big(\mathrm{C}_\infty(Y), \mathrm{C}_\infty(X)\big)$ is obtained in this way from some (uniquely determined) continuous map $\Psi_* : X \to Y$.

4.4. **Representations of C*-algebras.** Before continuing with examples of morphisms of C*-algebras let us give one more definition.

Definition 4.17. Let A be a C*-algebra and let H be a Hilbert space. A *representation* of A on H is a $*$-homomorphism $\pi : A \to \mathrm{B}(H)$. A representation π is *non degenerate* if the set

$$\big\{\pi(a)\xi \big| a \in A,\ \xi \in H\big\} \tag{4.8}$$

is dense in H. We say that a representation π is faithful if π is an injective map.

A representation which fails to be non degenerate is called "degenerate". The following Lemma gives an alternative description of the notion of a non degenerate representation.

Lemma 4.18. *Let A be a C*-algebra and let π be a representation of A on a Hilbert space H. Then π is non degenerate if and only if for any non zero $\eta \in H$ there exists $a \in A$ such that $\pi(a)\eta \neq 0$.*

Proof. Assume that π is non degenerate and for some $\eta \in H$ we have $\pi(a^*)\eta = 0$ for all $a \in A$. Then for any $\xi \in H$

$$0 = (\xi|\pi(a^*)|\eta) = (\pi(a)\xi|\eta)\,,$$

so η is orthogonal to a linearly dense set and in conclusion $\eta = 0$.

Now let π be any representation of A on H (possibly degenerate) and let K be the closure of $\big\{\pi(a)\xi \big| a \in A,\ \xi \in H\big\}$. If $K \neq H$ then there is a non zero $\eta \in K^\perp$. Then for all $\xi \in H$ and all $a \in A$ we have

$$0 = (\eta|\pi(a^*)|\xi) = (\pi(a)\eta|\xi)\,.$$

Thus $\pi(a)\eta$ is orthogonal to H for any $a \in A$, which means that $\pi(a)\eta = 0$ for all $a \in A$. ☐

Remark 4.19. It is interesting to note that the, so called, *Cohen's factorization theorem* implies that if the set (4.8) is dense in H then it is, in fact, equal to H. A particularly simple proof of this for σ-unital C*-algebras is due to S.L. Woronowicz.

Example 4.20. Let A be a C*-algebra and let H be a Hilbert space. Then every element $\pi \in \mathrm{Mor}\big(A, \mathcal{K}(H)\big)$ defines a non degenerate representation of A on H. Conversely any non degenerate representation π of A on H is an element of $\mathrm{Mor}\big(A, \mathcal{K}(H)\big)$.

The first statement is trivial, since $\mathrm{M}\big(\mathcal{K}(H)\big) = \mathrm{B}(H)$. For the second statement we must show that if π is a non degenerate representation of A on H then $\pi(A)\,\mathcal{K}(H)$ is dense in $\mathcal{K}(H)$. This is also clear: take $\xi, \eta \in H$. It is enough to show that $|\xi\,)\,(\,\eta|$ is in the closure of the set $\pi(A)\,\mathcal{K}(H)$. Now since π is non degenerate, for any $\varepsilon > 0$ there exist $\xi_\varepsilon \in H$ and $a_\varepsilon \in A$ such that $\big\|\pi(a_\varepsilon)\xi_\varepsilon - \xi\big\| < \varepsilon$. Therefore

$$\big\|\pi(a_\varepsilon)\,|\xi_\varepsilon\,)\,(\,\eta| - |\xi\,)\,(\,\eta|\big\| < \varepsilon\|\eta\|$$

which proves that $|\xi\,)\,(\,\eta|$ is in the closure of $\pi(A)\,\mathcal{K}(H)$.

4.5. **One more look at multipliers.** We have defined multiplier algebras in an abstract setting. Here we will show how to identify the multiplier algebra of a C*-algebra represented on a Hilbert space.

First let us note that if A is a C*-algebra and π is a non degenerate representation of A on a Hilbert space H then $\pi \in \mathrm{Mor}\bigl(A, \mathcal{K}(H)\bigr)$. Therefore the map π extends to a (unital) representation of the C*-algebra $\mathrm{M}(A)$, i.e. a $*$-homomorphism from $\mathrm{M}(A)$ to $\mathrm{B}(H)$. Furthermore, if π is faithful then the map $\mathrm{M}(A) \to \mathrm{B}(H)$ is injective. We can therefore identify $\mathrm{M}(A)$ with a certain set of operators acting on H. The next proposition gives a concrete description of this set.

In what follows we shall say that a C*-algebra B contained in $\mathrm{B}(H)$ for some Hilbert space H is acting non degenerately on H if the inclusion map $B \hookrightarrow \mathrm{B}(H)$ is a non degenerate representation of B.

Proposition 4.21. *Let H be a Hilbert space and let B be a C*-algebra of operators on a Hilbert space H. Assume that B is acting non degenerately on H. Then*

$$\mathrm{M}(B) = \bigl\{ m \in \mathrm{B}(H) \bigm| mb, bm \in A \text{ for all } b \in B \bigr\}.$$

Proof. Since B is an ideal in $\mathrm{M}(B)$, we see that any multiplier m of B must have the property that $mb, bm \in B$ for all $b \in B$. It remains to show that every bounded operator m on H with this property defines a multiplier of B. This is quite obvious: if $mb, bm \in B$ for all $b \in B$ then the map $B \ni b \mapsto mb$ is a multiplier of B, since

$$b^* m b' = (m^* b)^* b'$$

for all $b, b' \in B$. We have already seen that the map from the abstractly defined C*-algebra $\mathrm{M}(B)$ into $\mathrm{B}(H)$ is injective, so this multiplier is realized as left multiplication by a unique operator in $\mathrm{B}(H)$. Therefore this operator must be m. ☐

Proposition 4.21 provides a method of determining the multiplier algebra of the C*-algebra $\mathrm{C}_\infty(X)$ for a locally compact space X (Example 4.9). It is enough to represent $\mathrm{C}_\infty(X)$ faithfully and non degenerately on a Hilbert space and then $\mathrm{M}\bigl(\mathrm{C}_\infty(X)\bigr)$ can be easily found. The easiest (but usually not the most economical) way is to take $H = \ell^2(X)$ and represent $\mathrm{C}_\infty(X)$ by multiplication operators.

4.6. **Representations of groups in C*-algebras.**

Definition 4.22. Let G be a topological group and let B be a C*-algebra. A *representation* of G in B is a map $U : G \ni t \mapsto U_t \in \mathrm{M}(B)$ such that

(1) U_t is unitary for all $t \in G$,
(2) for any $t, s \in G$ we have $U_t U_s = U_{ts}$,
(3) for any $b \in B$ the map $G \ni t \mapsto U_t b \in B$ is continuous.

The set of all representations of G in B will be denoted by the symbol $\mathrm{Rep}(G, B)$.

Proposition 4.23.

(1) *Let G be a topological group and let $\mathcal{H}$ be a Hilbert space. Then each $U \in \mathrm{Rep}\bigl(G, \mathcal{K}(\mathcal{H})\bigr)$ is a strongly continuous unitary representation of G on $\mathcal{H}$.*
(2) *Conversely, any strongly continuous unitary representation of G on $\mathcal{H}$ defines an element $U \in \mathrm{Rep}\bigl(G, \mathcal{K}(\mathcal{H})\bigr)$.*

Proof. Ad (1). Clearly $t \mapsto U_t$ is a homomorphism from G into the unitary group of $\mathrm{B}(\mathcal{H}) = \mathrm{M}\bigl(\mathcal{K}(\mathcal{H})\bigr)$. Take $x \in \mathcal{H}$. For any $y \in \mathcal{H}$ the operator $|x)(y|$ belongs to $\mathcal{K}(\mathcal{H})$. Therefore if for $t \in G$ we define $T_t = U_t \circ |x)(y| - |x)(y|$ then

$$\|T_t\| \xrightarrow[t \to e]{} 0. \tag{4.9}$$

On the other hand

$$\|T_t\| = \sup_{\|z\|=1} \|T_t z\| = \sup_{\|z\|=1} \bigl\| (y|z)\,(U_t x - x) \bigr\| = \|y\| \|U_t x - x\|, \tag{4.10}$$

so by (4.9) we have $\|U_t x - x\| \xrightarrow[t \to e]{} 0$.

Ad (2). If for any $x \in \mathcal{H}$ we have $\|U_t x - x\| \xrightarrow[t \to e]{} 0$ then the computation (4.10) shows that we in fact have (4.9) and it remains true when $|x\,)\,(\,y|$ is replaced by any finite rank operator. Now any compact operator m is a limit of a sequence of finite rank operators. Let $\varepsilon > 0$ be given. There exists a finite rank operator c such that $\|m - c\| < \frac{\varepsilon}{3}$. We have

$$\|U_t m - m\| \leq \|U_t m - U_t c\| + \|U_t c - c\| + \|c - m\| < \tfrac{2}{3}\varepsilon + \|U_t c - c\|$$

which is smaller than ε for t in a sufficiently small neighborhood of e. ☐

4.7. Group actions on C*-algebras.

Definition 4.24. Let G be a topological group and let A be a C*-algebra. A map

$$\alpha : G \ni t \mapsto \alpha_t \in \mathrm{Aut}(A)$$

is called an *action* of G on A if

1. for any $t, s \in G$ we have $\alpha_t \circ \alpha_s = \alpha_{ts}$,
2. for any $a \in A$ the map $G \ni t \mapsto \alpha_t(a) \in A$ is continuous.

The set of all actions of G on A will be denoted by the symbol $\mathrm{Act}(G, A)$.

When G is a locally compact group and α is an action of G on A then we say that (A, G, α) is a C*-*dynamical system.*

Proposition 4.25. *Let G be a topological group and let A be a C*-algebra. Let $U \in \mathrm{Rep}(G, A)$. Define $\alpha : G \ni t \mapsto \alpha_t \in \mathrm{Aut}(A)$ by*

$$\alpha_t(a) = U_t a U_t^*.$$

Then $\alpha \in \mathrm{Act}(G, A)$.

Proof. It is easy to see that $t \mapsto \alpha_t$ is a homomorphism from G into $\mathrm{Aut}(B)$. We must prove that it is appropriately continuous.

We shall extensively use the fact that for any unitary element u of $\mathrm{M}(A)$ and any $x \in A$ we have $\|xu\| = \|x\| = \|ux\|$ (proof uses the C*-identity). Take $t, s \in G$ and $b \in B$.

$$\begin{aligned}
\big\|\alpha_t(b) - \alpha_s(b)\big\| &= \|U_t b U_t^* - U_s b U_s^*\| \\
&= \|U_t b U_{t^{-1}s} - U_s b\| \\
&\leq \|U_t b U_{t^{-1}s} - U_s b U_{t^{-1}s}\| + \|U_s b U_{t^{-1}s} - U_s b\| \\
&= \|U_t b - U_s b\| + \|b U_{t^{-1}s} - b\| \\
&= \|U_t b - U_s b\| + \|b U_{t^{-1}} - b U_{s^{-1}}\| \\
&= \|U_t b - U_s b\| + \|b U_t^* - b U_s^*\| \\
&= \|U_t b - U_s b\| + \|U_t b^* - U_s b^*\|.
\end{aligned}$$

Now since $\|U_t y - U_s y\| \xrightarrow[t \to s]{} 0$ for any $y \in B$ (because $U \in \mathrm{Rep}(G, B)$) we see that $\alpha \in \mathrm{Act}(G, B)$. ☐

Notation 4.26. Let G be a topological group and let A be a C*-algebra. Let $U \in \mathrm{Rep}(G, A)$. Then the action $\alpha \in \mathrm{Act}(G, A)$ defined in Proposition 4.25 will be denoted by the symbol Ad_U.

An automorphism of a C*-algebra given by conjugation with a unitary multiplier is called *inner.* Of course if A is a C*-algebra, G a topological group and and $U \in \mathrm{Rep}(G, A)$ then Ad_U is an action of G on A by inner automorphisms. Such an action is called an *inner action.*

Example 4.27. Let X be a locally compact space and let G be a topological group acting on X. This means that we have a continuous map

$$G \times X \ni (t, x) \longmapsto tx \in X$$

such that for each t the map $x \mapsto tx$ is a homeomorphism $X \to X$ and $t(sx) = (ts)x$ for all $t, s \in G$, $x \in X$.

There is a naturally associated action α of G on the C*-algebra $\mathrm{C}_\infty(X)$ defined by

$$\big(\alpha_t(f)\big)(x) = f(t^{-1}x) \tag{4.11}$$

for all $f \in \mathrm{C}_\infty(X)$, $t \in G$ and $x \in X$. Moreover, any action α of G on $\mathrm{C}_\infty(X)$ arises in this way from a unique action of G on X.

Notice that any group action on a commutative C*-algebra is inner if and only if it is trivial.

5. Crossed products

5.1. Definition and uniqueness.

Definition 5.1. Let (A, G, α) be a C*-dynamical system.

(1) A *covariant representation* of (A, G, α) in a C*-algebra B is a pair (π, U), where $\pi \in \mathrm{Mor}(A, B)$ and $U \in \mathrm{Rep}(G, B)$ are such that for any $a \in A$ and $t \in G$
$$\pi\big(\alpha_t(a)\big) = U_t \pi(a) U_t^*.$$

(2) A *crossed product* of A by (the action α of) G is a C*-algebra C with a covariant representation (π, U) of (A, G, α) in C such that for any C*-algebra B and any covariant representation (ρ, V) of (A, G, α) in B there exists a unique $\Phi \in \mathrm{Mor}(C, B)$ such that $\rho = \Phi \circ \pi$ and $V_t = \Phi(U_t)$ for all $t \in G$.

Proposition 5.2. *Let (A, G, α) be a C*-dynamical system. Then for any two crossed products (C, π, U) and (C', π', U') of A by G there is an isomorphism $\Psi \in \mathrm{Mor}(C, C')$ such that $\pi' = \Psi \circ \pi$ and $U'_t = \Psi(U_t)$ for all $t \in G$.*

Proof. By the definition of crossed product there is $\Psi \in \mathrm{Mor}(C, C')$ satisfying the conditions of the proposition save, perhaps, for being an isomorphism. A corresponding $\Psi' \in \mathrm{Mor}(C', C)$ also exists. Moreover their compositions $\Psi \circ \Psi'$ and $\Psi' \circ \Psi$ fulfill the conditions for the unique element of $\mathrm{Mor}(C', C')$ and $\mathrm{Mor}(C, C)$ respectively which do not interfere with neither π' nor U' (π and U respectively). But the identity morphisms on C' and C are such morphisms and they are, by definition of crossed product, unique. It follows that Ψ is an isomorphism. ☐

Let (A, G, α) be a C*-dynamical system. Proposition 5.2 shows that a crossed product of A by G is unique up to isomorphism preserving the covariant representation of (A, G, α) in the crossed product. We shall denote the crossed product of A by the action α of G by the symbol $A \rtimes_\alpha G$. The associated covariant representation of (A, G, α) in $A \rtimes_\alpha G$ will be denoted by $(\boldsymbol{\pi}, \boldsymbol{U})$.

The property of $(A \rtimes_\alpha G, \boldsymbol{\pi}, \boldsymbol{U})$ that for any C*-algebra B and any covariant representation of (A, G, α) in B there exists a unique $\Phi \in \mathrm{Mor}(A \rtimes_\alpha G, B)$ such that
$$\begin{aligned} \pi &= \Phi \circ \boldsymbol{\pi}, \\ U_t &= \Phi(\boldsymbol{U}_t) \end{aligned} \tag{5.1}$$
for all $t \in G$ is called the *universal property* defining the crossed product.

5.2. Existence.

Let (A, G, α) be a C*-dynamical system. We shall put normed $*$-algebra structure on the space $\mathrm{C_c}(G, A)$ consisting of all continuous functions $G \to A$ with compact support. Let us fix a left Haar measue h^L on G and let Δ be the modular function of G. For $\mathfrak{F}, \mathfrak{G} \in \mathrm{C_c}(G, A)$ and $t \in G$ we set
$$\left.\begin{aligned} (\mathfrak{F} \star \mathfrak{G})(t) &= \int_G \mathfrak{F}(s) \alpha_s\big(\mathfrak{G}(s^{-1}t)\big)\, dh^L(s), \\ \mathfrak{F}^*(t) &= \Delta(t)^{-1} \alpha_t\big(\mathfrak{F}(t^{-1})^*\big), \\ \|\mathfrak{F}\|_1 &= \int_G \big\|\mathfrak{F}(s)\big\|\, dh^L(s). \end{aligned}\right\} \tag{5.2}$$
The integrals in these formulas are well defined. In both cases we integrate continuous functions over compact sets. In the last formula this is obvious. In the first formula the integration is over $t(\operatorname{supp}\mathfrak{G})^{-1} \cap \operatorname{supp}\mathfrak{F}$ which is compact.

Lemma 5.3. *Let (A, G, α) be a C*-dynamical system. Then (5.2) endows the vector space $\mathrm{C_c}(G, A)$ with a normed $*$-algebra structure. In other words*

(1) *for any $\mathfrak{F}, \mathfrak{G} \in \mathrm{C_c}(G, A)$ the function $\mathfrak{F} \star \mathfrak{G}$ belongs to $\mathrm{C_c}(G, A)$,*

(2) *for any* $\mathfrak{F} \in \mathrm{C_c}(G, A)$ *the function* $\mathfrak{F}^*$ *belongs to* $\mathrm{C_c}(G, A)$,
(3) *the map* $\mathrm{C_c}(G, A)^2 \ni (\mathfrak{F}, \mathfrak{G}) \mapsto \mathfrak{F} \star \mathfrak{G}$ *is bilinear,*
(4) $(\mathfrak{F} \star \mathfrak{G}) \star \mathfrak{H} = \mathfrak{F} \star (\mathfrak{G} \star \mathfrak{H})$ *for all* $\mathfrak{F}, \mathfrak{G}, \mathfrak{H} \in \mathrm{C_c}(G, A)$,
(5) $(\mathfrak{F} \star \mathfrak{G})^* = \mathfrak{F}^* \star \mathfrak{G}^*$ *for all* $\mathfrak{F}, \mathfrak{G} \in \mathrm{C_c}(G, A)$,
(6) $\|\cdot\|_1$ *is a norm on the vector space* $\mathrm{C_c}(G, A)$,
(7) $\|\mathfrak{F} \star \mathfrak{G}\|_1 \leq \|\mathfrak{F}\|_1 \|\mathfrak{G}\|_1$ *for all* $\mathfrak{F}, \mathfrak{G} \in \mathrm{C_c}(G, A)$,
(8) $\|\mathfrak{F}^*\|_1 = \|\mathfrak{F}\|_1$ *for all* $\mathfrak{F} \in \mathrm{C_c}(G, A)$.

Proof. (2), (3) and (6) are obvious. For (8) we use Proposition 3.4 (2).

Ad (1). The integral defining $\mathfrak{F} \star \mathfrak{G}$ is non zero only if $t(\operatorname{supp} \mathfrak{F})^{-1} \cap \operatorname{supp} \mathfrak{G} \neq \emptyset$. Therefore for $(\mathfrak{F} \star \mathfrak{G})(t)$ to be non zero it is necessary that $t \in \operatorname{supp} \mathfrak{F} \operatorname{supp} \mathfrak{G}$ which is compact ($r \in \operatorname{supp} \mathfrak{F}$ belongs to $t(\operatorname{supp} \mathfrak{G})^{-1}$ if and only if there is $s \in \operatorname{supp} \mathfrak{G}$ such that $ts^{-1} = r$, i.e. $t = rs$). The continuity of $\mathfrak{F} \star \mathfrak{G}$ follows from the fact that the functions

$$G \ni s \mapsto \mathfrak{G}(s^{-1}t') \in A$$

converge uniformly to $s \mapsto \mathfrak{G}(s^{-1}t)$ as $t' \to t$, i.e. from the uniform continuity of compactly supported functions (Proposition 3.3).

Ad (4). Here we need to use continuity of α, twice Fubini's theorem and once the left invariance of h^L:

$$\begin{aligned}
\big((\mathfrak{F} \star \mathfrak{G}) \star \mathfrak{H}\big)(t) &= \int_G (\mathfrak{F} \star \mathfrak{G})(s)\alpha_s\big(\mathfrak{H}(s^{-1}t)\big)\, dh^l(s) \\
&= \int_G \left(\int_G \mathfrak{F}(r)\alpha_r\big(\mathfrak{G}(r^{-1}s)\big)\, dh^L(r) \right) \alpha_s\big(\mathfrak{H}(s^{-1}t)\big)\, dh^L(s) \\
&= \int_{G\times G} \mathfrak{F}(r)\alpha_r\big(\mathfrak{G}(r^{-1}s)\big)\alpha_s\big(\mathfrak{H}(s^{-1}t)\big)\, d(h^L \otimes h^L)(r, s) \\
&= \int_{G\times G} \mathfrak{F}(r)\alpha_r\Big(\mathfrak{G}(r^{-1}s)\alpha_{r^{-1}s}\big(\mathfrak{H}(s^{-1}t)\big)\Big)\, d(h^L \otimes h^L)(r, s) \\
&= \int_G \mathfrak{F}(r)\alpha_r \left(\int_G \mathfrak{G}(r^{-1}s)\alpha_{r^{-1}s}\big(\mathfrak{H}(s^{-1}t)\big)\, dh^L(s) \right) dh^L(r) \\
&= \int_G \mathfrak{F}(r)\alpha_r \left(\int_G \mathfrak{G}(s)\alpha_s\big(\mathfrak{H}(s^{-1}r^{-1}t)\big)\, dh^L(s) \right) dh^L(r) \\
&= \int_G \mathfrak{F}(r)\alpha_r\big((\mathfrak{G} \star \mathfrak{H})(r^{-1}t)\big)\, dh^L(r) = \big(\mathfrak{F} \star (\mathfrak{G} \star \mathfrak{H})\big)(t).
\end{aligned}$$

Ad (5). Remembering that Δ is a homomorphism we compute:

$$\begin{aligned}
(\mathfrak{F}^*\star\mathfrak{G}^*)(t)&=\int\limits_G \mathfrak{F}^*(s)\alpha_s\big(\mathfrak{G}^*(s^{-1}t)\big)\,dh^L(s)\\
&=\int\limits_G \Delta(s)^{-1}\alpha_s\big((\mathfrak{F}(s^{-1})^*\big)\alpha_s\Big(\Delta(s^{-1}t)^{-1}\alpha_{s^{-1}t}\big(\mathfrak{G}(t^{-1}s)^*\big)\Big)\,dh^L(s)\\
&=\Delta(t)^{-1}\int\limits_G \alpha_s\big(\mathfrak{F}(s^{-1})^*\big)\alpha_t\big(\mathfrak{G}(t^{-1}s)^*\big)\,dh^L(s)\\
&=\Delta(t)^{-1}\alpha_t\left(\int\limits_G \alpha_{t^{-1}s}\big(\mathfrak{F}(s^{-1})\big)^*\mathfrak{G}(t^{-1}s)^*\,dh^L(s)\right)\\
&=\Delta(t)^{-1}\alpha_t\left(\left[\int\limits_G \mathfrak{G}(t^{-1}s)\alpha_{t^{-1}s}\big(\mathfrak{F}(s^{-1})\big)\,dh^L(s)\right]^*\right)\\
&=\Delta(t)^{-1}\alpha_t\left(\left[\int\limits_G \mathfrak{G}(s)\alpha_s\big(\mathfrak{F}(s^{-1}t^{-1})\big)\,dh^L(s)\right]^*\right)\\
&=\Delta(t)^{-1}\alpha_t\big(\big[(\mathfrak{G}\star\mathfrak{F})(t)\big]^*\big)=(\mathfrak{G}\star\mathfrak{F})^*(t).
\end{aligned}$$

Ad (7). Since

$$\|\mathfrak{F}\star\mathfrak{G}\|_1=\int\limits_G\left\|\int\limits_G \mathfrak{F}(s)\alpha_s\big(\mathfrak{G}(s^{-1}t)\big)\,dh^L(s)\right\|\,dh^L(t)\leq\int\limits_G\int\limits_G\big\|\mathfrak{F}(s)\big\|\big\|\mathfrak{G}(s^{-1}t)\big\|\,dh^L(s)\,dh^L(t),$$

we see that the desired inequality follows from the classical inequality

$$\|f\star g\|_1\leq\|f\|_1\|g\|_1,$$

(with "$\star$" and $\|\cdot\|_1$ denoting the usual convolution and norm of L^1 functions on G) for $f(t)=\big\|\mathfrak{F}(t)\big\|$ and $g(t)=\big\|\mathfrak{G}(t)\big\|$ (cf. Lemma 3.5). ☐

Definition 5.4. Let (A,G,α) be a C*-dynamical system. The product "$\star$" on $\mathrm{C_c}(G,A)$ defined in (5.2) is called the *convolution product*. The normed $*$-algebra $\mathrm{C_c}(G,A)$ is called the *twisted convolution algebra* of the dynamical system (A,G,α).

Remark 5.5. Let (A,G,α) be a C*-dynamical system. The name "twisted convolution algebra" is in literature used also for the *completion* of $\mathrm{C_c}(G,A)$ in the norm $\|\cdot\|_1$. The Banach $*$-algebra obtained this way is denoted by $L^1(G,A)$.

Let (A,G,α) be a C*-dynamical system and let us endow $\mathrm{C_c}(G,A)$ with the normed $*$-algebra structure described by formulas (5.2). The fact that $\mathrm{C_c}(G,A)$ is a *normed* $*$-algebra will not be essential at all for our purposes. We will only need the obvious fact that for each $\mathfrak{F}\in\mathrm{C_c}(G,A)$ the quantity $\|\mathfrak{F}\|_1$ is finite.

Lemma 5.6. *Let (A,G,α) be a C*-dynamical system and let $\mathrm{C_c}(G,A)$ be the associated twisted convolution algebra. Let B be a C*-algebra and let (π,U) be a covariant representation of (A,G,α). Then the formula*

$$(\pi\rtimes_\alpha U)(\mathfrak{F})=\int\limits_G \pi\big(\mathfrak{F}(t)\big)U_t\,dh^L(t)$$

defines a $*$-homomorphism $\pi\rtimes_\alpha U:\mathrm{C_c}(G,A)\to\mathrm{M}(B)$ such that

(1) $(\pi\rtimes_\alpha U)\big(\mathrm{C_c}(G,A)\big)B$ *is dense in* B,
(2) $\big\|(\pi\rtimes_\alpha U)(\mathfrak{F})\big\|\leq\|\mathfrak{F}\|_1$.

Proof. Let us first check that $\pi \rtimes_\alpha U$ is a $*$-homomorphism:

$$\begin{aligned}
(\pi \rtimes_\alpha U)(\mathfrak{F} \star \mathfrak{G}) &= \int_G \pi\big((\mathfrak{F} \star \mathfrak{G})(t)\big) U_t \, dh^L(t) \\
&= \int_G \pi \left(\int_G \mathfrak{F}(s) \alpha_s\big(\mathfrak{G}(s^{-1}t)\big) \right) U_t \, dh^L(t) \\
&= \int_{G \times G} \pi\big(\mathfrak{F}(s)\big) \pi\Big(\alpha_s\big(\mathfrak{G}(s^{-1}t)\big)\Big) U_t \, d(h^L \otimes h^L)(t,s) \\
&= \int_{G \times G} \pi\big(\mathfrak{F}(s)\big) U_s \pi\big(\mathfrak{G}(s^{-1}t)\big) U_{s^{-1}t} \, d(h^L \otimes h^L)(t,s) \\
&= \int_G \pi\big(\mathfrak{F}(s)\big) U_s \left(\int_G \pi\big(\mathfrak{G}(s^{-1}t)\big) U_{s^{-1}t} \, dh^L(t) \right) dh^L(s) \\
&= \int_G \pi\big(\mathfrak{F}(s)\big) U_s \left(\int_G \pi\big(\mathfrak{G}(t)\big) U_t \, dh^L(t) \right) dh^L(s) \\
&= \int_G \pi\big(\mathfrak{F}(s)\big) U_s \, dh^L(s) \int_G \pi\big(\mathfrak{G}(t)\big) U_t \, dh^L(t) = (\pi \rtimes_\alpha U)(\mathfrak{F})(\pi \rtimes_\alpha U)(\mathfrak{G}),
\end{aligned}$$

and

$$\begin{aligned}
\big((pi \rtimes_\alpha U)(\mathfrak{F})\big)^* &= \left(\int_G \pi\big(\mathfrak{F}(t)\big) U_t \, dh^L(t) \right)^* \\
&= \int_G U_t^* \pi\big(\mathfrak{F}(t)^*\big) \, dh^L(t) \\
&= \int_G U_{t^{-1}} \pi\big(\mathfrak{F}(t)^*\big) U_{t^{-1}}^* U_{t^{-1}} \, dh^L(t) \\
&= \int_G \pi\Big(\alpha_{t^{-1}}\big(\mathfrak{F}(t)^*\big)\Big) U_{t^{-1}} \, dh^L(t) \\
&= \int_G \Delta(t)^{-1} \pi\Big(\alpha_t\big(\mathfrak{F}(t^{-1})^*\big)\Big) U_t \, dh^L(t) \\
&= \int_G \pi\Big(\Delta(t)^{-1} \alpha_t\big(\mathfrak{F}(t^{-1})^*\big)\Big) U_t \, dh^L(t) \\
&= \int_G \pi\big(\mathfrak{F}^*(t)\big) U_t \, dh^L(t) = (\pi \rtimes_\alpha U)(\mathfrak{F}^*)
\end{aligned}$$

(cf. Proposition 3.4 (2)).

Ad (1). Take $a \in A$ and $b \in B$. Since $\pi(A)B$ is dense in B, it is enough to show that $\pi(a)b$ is in the closure of $(\pi \rtimes_\alpha U)\big(\mathrm{C_c}(G,A)\big)B$. To that end let $(f_\lambda)_{\lambda \in \Lambda}$ be a net of positive continuous functions on G with compact support and such that $\operatorname{supp} f_\lambda$ is a directed descending family with $\bigcap_{\lambda \in \Lambda} \operatorname{supp} f_\lambda = \{e\}$ and $\int_G f_\lambda(t) \, dh^L(t) = 1$ for all $\lambda \in \Lambda$. Let $\mathfrak{F}_\lambda(t) = f_\lambda(t) \alpha_{t^{-1}}(a)$. Then $(\mathfrak{F}_\lambda)_{\lambda \in \Lambda}$ is a net in $\mathrm{C_c}(G,A)$ and

$$(\pi \rtimes_\alpha U)(\mathfrak{F}_\lambda) = \int_G \pi\big(\mathfrak{F}(t)\big) U_t \, dh^L(t) = \int_G U_t \pi\Big(\alpha_t\big(\mathfrak{F}(t)\big)\Big) \, dh^L(t) = \int_G f_\lambda(t) U_t \, dh^L(t) \, \pi(a) b$$

which converges to $\pi(a)b$.

Statement (2) is immediate. □

In the next proposition we shall use the fact that for any Hilbert space $\mathcal{H}$ the Hilbert space tensor product $\mathcal{H}\otimes L^2(G)$ can be identified with the Hilbert space $L^2(G,\mathcal{H})$. This last Hilbert space can be viewed as the completion of the space $\mathrm{C_c}(G,\mathcal{H})$ in the norm coming from the scalar product

$$(\psi|\xi)=\int\limits_G\big(\psi(t)|\xi(t)\big)\,dh^L(t)$$

for $\psi,\xi\in\mathrm{C_c}(G,\mathcal{H})$.

Proposition 5.7. *Let (A,G,α) be a C^*-dynamical system. Let π_0 be a non degenerate representation of A on a Hilbert space $\mathcal{H}$. Let $H=\mathcal{H}\otimes L^2(G)$. Define $\pi:A\to\mathrm{B}(H)$ and $U:G\to\mathrm{B}(H)$ by*

$$\big(\pi(a)\xi\big)(s)=\pi_0\big(\alpha_{s^{-1}}(a)\big)\xi(s),$$
$$\big(U_t\xi\big)(s)=\xi(t^{-1}s)$$

for all $s,t\in G$ and all $\xi\in\mathrm{C_c}(G,\mathcal{H})$. Then

1. *$\pi\in\mathrm{Mor}\big(A,\mathcal{K}(H)\big)$,*
2. *$U\in\mathrm{Rep}\big(G,\mathcal{K}(H)\big)$,*
3. *(π,U) is a covariant representation of (A,G,α) in $\mathcal{K}(H)$,*
4. *if π_0 is faithful then $\pi\rtimes_\alpha U$ is injective.*

Proof. Statement (2) is obvious. As for (1) let us see that for any non zero $\xi\in\mathrm{C_c}(G,\mathcal{H})$ there is $a\in A$ such that $\pi(a)\xi\neq0$. Indeed, if $\xi\neq0$ then there is $t\in G$ such that $\xi(t)\neq0$. Since π_0 is non degenerate, by Lemma 4.18, there is an element $b\in A$ such that $b\xi(t)\neq0$. Now let $a=\alpha_t(b)$. Then $\big(\pi(a)\xi\big)(t)=\pi_0(b)\xi(t)\neq0$ and by continuity this function is non zero in a neighborhood of t. Therefore if is non zero as an element of $H=\mathcal{H}\otimes L^2(G)$.

For a general non zero $\eta\in\mathcal{H}\otimes L^2(G)$ we approximate η by elements of $\mathrm{C_c}(G,H)$ and find an element $a\in A$ such that $\pi(a)\eta\neq0$. It follows from Lemma 4.18 that π is a non degenerate representation of A on H. In other words $\pi\in\mathrm{Mor}\big(A,\mathcal{K}(H)\big)$.

Ad (3). Take $\xi\in\mathrm{C_c}(G,\mathcal{H})$, $a\in A$ and $t\in G$. Then for any $s\in G$ we have

$$\begin{aligned}\big(U_t\pi(a)U_t^*\xi\big)(s)&=\big(\pi(a)U_t\xi\big)(t^{-1}s)\\&=\pi_0\big(\alpha_{s^{-1}t}(a)\big)\big((U_t^*\xi)(t^{-1}s)\big)\\&=\pi_0\Big(\alpha_{s^{-1}}\big(\alpha_t(a)\big)\Big)\xi(s)=\Big(\pi\big(\alpha_t(a)\big)\xi\Big)(s).\end{aligned}$$

Ad (4). Let $\mathfrak{F}\in\mathrm{C_c}(G,A)$ be non zero. There is $r\in G$ such that $\mathfrak{F}(r)\neq0$. Since π_0 is injective there are vectors $\xi_0,\eta_0\in\mathcal{H}$ such that

$$\big(\eta_0\big|\pi_0\big(\mathfrak{F}(r)\big)\big|\xi_0\big)\neq0. \tag{5.3}$$

We will produce two vectors $\xi,\eta\in H$ such that

$$\big|(\eta|(\pi\rtimes_\alpha U)(\mathfrak{F})|\xi)-\big(\eta_0\big|\pi_0\big(\mathfrak{F}(r)\big)\big|\xi_0\big)\big|<\big|\big(\eta_0\big|\pi_0\big(\mathfrak{F}(r)\big)\big|\xi_0\big)\big|.$$

This means that $(\eta|(\pi\rtimes_\alpha U)(\mathfrak{F})|\xi)\neq0$, so $(\pi\rtimes_\alpha U)(\mathfrak{F})\neq0$.

Since $(t,s)\mapsto\Big(\alpha_{s^{-1}}\big(\mathfrak{F}(t)\big)-\mathfrak{F}(r)\Big)$ is a continuous function, there is a neighborhood $\mathcal{V}$ of (r,e) such that for $(t,s)\in\mathcal{V}$ we have

$$\Big\|\alpha_{s^{-1}}\big(\mathfrak{F}(t)\big)-\mathfrak{F}(r)\Big\|<\frac{\big|\big(\eta_0\big|\pi_0\big(\mathfrak{F}(r)\big)\big|\xi_0\big)\big|}{\|\xi_0\|\|\eta_0\|}.$$

Now let f and g be functions on G such that

- f and g are positive,
- $f,g\in\mathrm{C_c}(G)$,
- $g(s)f(t^{-1}s)\neq0\;\Rightarrow\;(t,s)\in\mathcal{V}$,

- $\int_G g(s)f(t^{-1}s)\,d(h^L\otimes h^L)(t,s)=1.$

This means that f and g are positive continuous functions with compact s supports such that $\operatorname{supp} f$ is close to $\{r^{-1}\}$ and $\operatorname{supp} g$ is close to $\{e\}$, so that the support of the function $(t,s)\mapsto g(s)f(t^{-1}s)$ is contained in $\mathcal{V}$. The fourth condition is only matter of normalization.

We let $\xi(s)=f(s)\xi_0$ and $\eta(s)=g(s)\eta_0$. Now we will compute the left hand side of (5.3).

Step 1:

$$\begin{aligned}\big[(\pi\rtimes_\alpha U)(\mathfrak{F})\xi\big](s)&=\left[\int_G \pi\big(\mathfrak{F}(t)\big)U_t\,dh^L(t)\,\xi\right](s)\\&=\int_G\big[\pi\big(\mathfrak{F}(t)\big)U_t\xi\big](s)\,dh^L(t)\\&=\int_G\pi_0\Big(\alpha_{s^{-1}}\big(\mathfrak{F}(t)\big)\Big)(U_t\xi)(s)\,dh^L(t)\\&=\int_G\pi_0\Big(\alpha_{s^{-1}}\big(\mathfrak{F}(t)\big)\Big)\xi(t^{-1}s)\,dh^L(t)\\&=\int_G\pi_0\Big(\alpha_{s^{-1}}\big(\mathfrak{F}(t)\big)\Big)\xi_0 f(t^{-1}s)\,dh^L(t).\end{aligned}$$

Step 2:

$$\begin{aligned}(\eta|(\pi\rtimes_\alpha U)(\mathfrak{F})|\xi)&=\int_G\big(\eta(s)\big|\big[(\pi\rtimes_\alpha U)(\mathfrak{F})\xi\big](s)\big)\;dh^L(s)\\&=\int_G\left(\eta(s)\middle|\int_G\pi_0\Big(\alpha_{s^{-1}}\big(\mathfrak{F}(t)\big)\Big)\xi_0 f(t^{-1}s)\,dh^L(t)\right)\;dh^L(s)\\&=\int_G\left(g(s)\eta_0\middle|\int_G\pi_0\Big(\alpha_{s^{-1}}\big(\mathfrak{F}(t)\big)\Big)\xi_0 f(t^{-1}s)\,dh^L(t)\right)\;dh^L(s)\\&=\int_{G\times G}g(s)\left(\eta_0\middle|\pi_0\Big(\alpha_{s^{-1}}\big(\mathfrak{F}(t)\big)\Big)\xi_0\right)f(t^{-1}s)\,d(h^L\otimes h^L)(t,s).\end{aligned}$$

Step 3:

$$\begin{aligned}
&\bigl|(\eta|(\pi\rtimes_\alpha U)(\mathfrak{F})|\xi)-\bigl(\eta_0\big|\pi_0\bigl(\mathfrak{F}(r)\bigr)\big|\xi_0\bigr)\bigr|\\
&\qquad=\left|\int\limits_{G\times G}g(s)\,\Bigl(\eta_0\Big|\pi_0\Bigl(\alpha_{s^{-1}}\bigl(\mathfrak{F}(t)\bigr)\Bigr)\xi_0\Bigr)\,f(t^{-1}s)\,d(h^L\otimes h^L)(t,s)\right.\\
&\qquad\qquad\left.-\int\limits_{G\times G}g(s)f(t^{-1}s)\,d(h^L\otimes h^L)(t,s)\,\bigl(\eta_0\big|\pi_0\bigl(\mathfrak{F}(r)\bigr)\big|\xi_0\bigr)\right|\\
&\qquad=\left|\int\limits_{G\times G}g(s)\,\Bigl(\eta_0\Big|\pi_0\Bigl(\alpha_{s^{-1}}\bigl(\mathfrak{F}(t)\bigr)\Bigr)\xi_0\Bigr)\,f(t^{-1}s)\,d(h^L\otimes h^L)(t,s)\right.\\
&\qquad\qquad\left.-\int\limits_{G\times G}g(s)\,\bigl(\eta_0\big|\pi_0\bigl(\mathfrak{F}(r)\bigr)\big|\xi_0\bigr)\,f(t^{-1}s)\,d(h^L\otimes h^L)(t,s)\right|\\
&\qquad\leq\left|\int\limits_{G\times G}g(s)\,\Bigl(\eta_0\Big|\pi_0\Bigl(\alpha_{s^{-1}}\bigl(\mathfrak{F}(t)\bigr)-\mathfrak{F}(r)\Bigr)\xi_0\Bigr)\,f(t^{-1}s)\,d(h^L\otimes h^L)(t,s)\right|\\
&\qquad\leq\|\xi_0\|\|\eta_0\|\int\limits_{G\times G}g(s)f(t^{-1}s)\bigl\|\alpha_{s^{-1}}\bigl(\mathfrak{F}(t)\bigr)-\mathfrak{F}(r)\bigr\|\,d(h^L\otimes h^L)(t,s)
\end{aligned}$$

which is strictly smaller than $\bigl|\bigl(\eta_0\big|\pi_0\bigl(\mathfrak{F}(r)\bigr)\big|\xi_0\bigr)\bigr|$. □

Definition 5.8. Let (A,G,α) be a C*-dynamical system and let π_0 be a non degenerate representation of A on a Hilbert space $\mathcal{H}$. The covariant representation of (A,G,α) in $\mathcal{K}\bigl(\mathcal{H}\otimes L^2(G)\bigr)$ described in Proposition 5.7 is called the *regular* representation of (A,G,α) associated to π_0.

Let (A,G,α) be a C*-dynamical system. The construction of regular representations of (A,G,α) gives us many covariant representations. In particular, since there always is a faithful representation π_0 of A, we obtain a covariant representation (π,U) with injective $\pi\rtimes_\alpha U$.

Let us also remark that the construction of regular representation of (A,G,α) can be generalized in a way allowing to obtain representations in much more general C*-algebras (not only $\mathcal{K}(H)$ for some H). This requires using the language of *Hilbert* C*-*modules.*

Theorem 5.9. *Let (A,G,α) be a* C*-*dynamical system. Then the crossed product of A by G exists.*

Proof. The proof will be given in several steps:

(1) definition of the C*-algebra $A\rtimes_\alpha G$,
(2) definition of the elements $\boldsymbol{\pi}$ and $\boldsymbol{U}$ of the universal covariant representation $(\boldsymbol{\pi},\boldsymbol{U})$,
(3) establishing boundedness of $\boldsymbol{\pi}(a)$ for each $a\in A$,
(4) establishing boundedness of $\boldsymbol{U}_t$ for each $t\in G$,
(5) proof that for each $a\in A$ we have $\boldsymbol{\pi}(a)\in\mathrm{M}(A\rtimes_\alpha G)$,
(6) proof that for each $t\in G$ we have $\boldsymbol{U}_t\in\mathrm{M}(A\rtimes_\alpha G)$,
(7) proof that $\boldsymbol{\pi}\in\mathrm{Mor}(A,A\rtimes_\alpha G)$,
(8) proof that $\boldsymbol{U}\in\mathrm{Rep}(G,A\rtimes_\alpha G)$,
(9) proof that $(\boldsymbol{\pi},\boldsymbol{U})$ is a covariant representation of (A,G,α) in $A\rtimes_\alpha G$,
(10) proof of the universal property of $(A\rtimes_\alpha G,\boldsymbol{\pi},\boldsymbol{U})$.

Ad (1). We shall define a new norm on $\mathrm{C_c}(G,A)$. This new norm will be a C*-norm and so the completion of $\mathrm{C_c}(G,A)$ in this norm will be a C*-algebra.

For $\mathfrak{F}\in\mathrm{C_c}(G,A)$ let

$$\|\mathfrak{F}\|=\sup\bigl\{\bigl\|(\pi\rtimes_\alpha U)(\mathfrak{F})\bigr\|\,\big|\,(\pi,U)\text{ is a covariant representation of }(A,G,\alpha)\bigr\}\tag{5.4}$$

It is not possible to take supremum over a *class,* but the right hand side of the above equation is a supremum of a subclass of a set (namely $\mathbb{R}$), so it is a supremum of a sub*set* of $\mathbb{R}$.

Let us note the following facts:

- thanks to Lemma 5.6 (2) we have $\|\mathfrak{F}\| < \infty$ for all $\mathfrak{F} \in \mathrm{C_c}(G, A)$, in fact $\|\mathfrak{F}\| \leq \|\mathfrak{F}\|_1$ for all $\mathfrak{F} \in \mathrm{C_c}(G, A)$,
- (5.4) defines a seminorm on $\mathrm{C_c}(G, A)$,
- the seminorm $\|\cdot\|$ satisfies $\|\mathfrak{F}^*\mathfrak{F}\| = \|\mathfrak{F}\|$ for all $\mathfrak{F} \in \mathrm{C_c}(G, A)$,
- the seminorm $\|\cdot\|$ is in fact a norm.

The last statement is a direct consequence of the fact that there always exists a covariant representation (U, π) of (A, G, α) such that $\pi \rtimes_\alpha U$ is injective (cf. paragraph after Definition 5.8).

We let $A \rtimes_\alpha G$ be the completion of $\mathrm{C_c}(G, A)$ with respect to the norm defined by (5.4). As mentioned in the beginning of the proof, $A \rtimes_\alpha G$ is a C*-algebra. Moreover $\mathrm{C_c}(G, A)$ is a dense $*$-subalgebra of $A \rtimes_\alpha G$.

Ad (2). Let us now define the universal covariant representation $(\boldsymbol{U}, \boldsymbol{\pi})$ of (A, G, α) in $A \rtimes_\alpha G$. For $a \in A$ and $t \in G$ let

$$\begin{aligned}\big(\boldsymbol{\pi}(a)\mathfrak{F}\big)(s) &= a\mathfrak{F}(s)\\ (\boldsymbol{U}_t\mathfrak{F})(s) &= \alpha_t\big(\mathfrak{F}(t^{-1}s)\big)\end{aligned}$$

for any $\mathfrak{F} \in \mathrm{C_c}(G, A)$ and any $s \in G$. This defines $\boldsymbol{\pi}(a)$ and $\boldsymbol{U}_t$ as maps $\mathrm{C_c}(G, A) \to \mathrm{C_c}(G, A)$.

Ad (3). The map $\boldsymbol{\pi}(a)$ extends to continuous a map of $A \rtimes_\alpha G$ into itself because

$$\begin{aligned}\big\|(\pi \rtimes_\alpha U)\big(\boldsymbol{\pi}(a)\mathfrak{F}\big)\big\| &= \left\|\int_G \pi\big(a\mathfrak{F}(t)\big)U_t\,dh^L(t)\right\|\\ &\leq \|a\|\left\|\int_G \pi\big(\mathfrak{F}(t)\big)U_t\,dh^L(t)\right\| = \|a\|\big\|(\pi \rtimes_\alpha U)\big(\mathfrak{F}\big)\big\|\end{aligned}$$

for any covariant representation (π, U) of (A, G, α). Therefore

$$\big\|\boldsymbol{\pi}(a)\mathfrak{F}\big\| \leq \|a\|\|\mathfrak{F}\|$$

for any $\mathfrak{F} \in \mathrm{C_c}(G, A)$.

Ad (4). To see that $\boldsymbol{U}_t$ is continuous we note that

$$\begin{aligned}\big\|(\pi \rtimes_\alpha U)\big(\boldsymbol{U}_t\mathfrak{F}\big)\big\| &= \left\|\int_G \pi\Big(\alpha_t\big(\mathfrak{F}(t^{-1}s)\big)\Big)U_s\,dh^L(s)\right\|\\ &= \left\|\int_G \pi\Big(\alpha_t\big(\mathfrak{F}(s)\big)\Big)U_{ts}\,dh^L(s)\right\|\\ &= \left\|\int_G \pi\Big(\alpha_t\big(\mathfrak{F}(s)\big)\Big)U_tU_s\,dh^L(s)\right\|\\ &= \left\|\int_G U_t\pi\big(\mathfrak{F}(s)\big)U_s\,dh^L(s)\right\|\\ &= \left\|U_t\int_G \pi\big(\mathfrak{F}(s)\big)U_s\,dh^L(s)\right\|\\ &= \left\|\int_G \pi\big(\mathfrak{F}(s)\big)U_s\,dh^L(s)\right\| = \big\|(\pi \rtimes_\alpha U)(\mathfrak{F})\big\|\end{aligned}$$

for any covariant representation of (π, U) of (A, G, α), so for each $t \in G$ the operator $\boldsymbol{U}_t$ is an isometry of $A \rtimes_\alpha G$.

Ad (5). Checking that operators $\boldsymbol{\pi}(a)$ are multipliers is easy:

$$
\begin{aligned}
\big(\mathfrak{F}^* \star \boldsymbol{\pi}(a)\mathfrak{G}\big)(t) &= \int\limits_G \mathfrak{F}^*(s)\alpha_s\big(a\mathfrak{G}(s^{-1}t)\big)\,dh^L(s) \\
&= \int\limits_G \Delta(s)^{-1}\alpha_s\big(\mathfrak{F}(s^{-1})^*\big)\alpha_s(a)\alpha_s\big(\mathfrak{G}(s^{-1}t)\big)\,dh^L(s) \\
&= \int\limits_G \Delta(s)^{-1}\alpha_s\big(\mathfrak{F}(s^{-1})^*a\big)\alpha_s\big(\mathfrak{G}(s^{-1}t)\big)\,dh^L(s) \\
&= \int\limits_G \Delta(s)^{-1}\alpha_s\big(a^*\mathfrak{F}(s^{-1})\big)^*\alpha_s\big(\mathfrak{G}(s^{-1}t)\big)\,dh^L(s) \\
&= \int\limits_G \big(\boldsymbol{\pi}(a)\mathfrak{F}\big)^*(s)\alpha_s\big(\mathfrak{G}(s^{-1}t)\big)\,dh^L(s) = \big((\boldsymbol{\pi}(a^*)\mathfrak{F})^* \star \mathfrak{G}\big)(t).
\end{aligned}
$$

Since this holds on the dense subspace $\mathrm{C_c}(G, A)$, we have that $\boldsymbol{\pi}(a) \in \mathrm{M}(A \rtimes_\alpha G)$. Moreover $\boldsymbol{\pi}(a)^* = \boldsymbol{\pi}(a^*)$.

Ad (6). Take $t \in G$. We have

$$
\begin{aligned}
\big(\mathfrak{F}^* \star \boldsymbol{U}_t\mathfrak{G}\big)(u) &= \int\limits_G \mathfrak{F}^*(s)\alpha_s\big((\boldsymbol{U}_t\mathfrak{G})(s^{-1}u)\big)\,dh^L(s) \\
&= \int\limits_G \mathfrak{F}^*(s)\alpha_s\Big(\alpha_t\big(\mathfrak{G}(t^{-1}s^{-1}u)\big)\Big)\,dh^L(s) \\
&= \int\limits_G \Delta(s)^{-1}\alpha_s\big(\mathfrak{F}(s^{-1})^*\big)\alpha_s\Big(\alpha_t\big(\mathfrak{G}(t^{-1}s^{-1}u)\big)\Big)\,dh^L(s) \\
&= \int\limits_G \alpha_s\Big(\mathfrak{F}(s^{-1})^*\alpha_t\big(\mathfrak{G}(t^{-1}s^{-1}u)\big)\Big)\Delta(s)^{-1}\,dh^L(s).
\end{aligned}
$$

Now recall (Proposition 3.4 (3)) that $\Delta(s)^{-1}\,dh^L(s)$ is a right Haar measure, so

$$
\begin{aligned}
\big(\mathfrak{F}^* \star \boldsymbol{U}_t\mathfrak{G}\big)(u) &= \int\limits_G \alpha_s\Big(\mathfrak{F}(s^{-1})^*\alpha_t\big(\mathfrak{G}(t^{-1}s^{-1}u)\big)\Big)\Delta(s)^{-1}\,dh^L(s) \\
&= \int\limits_G \alpha_{st^{-1}}\Big(\mathfrak{F}(ts^{-1})^*\alpha_t\big(\mathfrak{G}(s^{-1}u)\big)\Big)\Delta(s)^{-1}\,dh^L(s) \\
&= \int\limits_G \Delta(s)^{-1}\alpha_{st^{-1}}\big(\mathfrak{F}(ts^{-1})^*\big)\alpha_s\big(\mathfrak{G}(s^{-1}u)\big)\,dh^L(s) \\
&= \int\limits_G \Delta(s)^{-1}\alpha_s\Big(\alpha_{t^{-1}}\big(\mathfrak{F}(ts^{-1})^*\big)\Big)\alpha_s\big(\mathfrak{G}(s^{-1}u)\big)\,dh^L(s) \\
&= \int\limits_G \Delta(s)^{-1}\alpha_s\Big(\big((\boldsymbol{U}_{t^{-1}}\mathfrak{F})(s^{-1})^*\big)\Big)\alpha_s\big(\mathfrak{G}(s^{-1}u)\big)\,dh^L(s) \\
&= \int\limits_G (\boldsymbol{U}_{t^{-1}}\mathfrak{F})^*(s)\alpha_s\big(\mathfrak{G}(s^{-1}u)\big)\,dh^L(s).
\end{aligned}
$$

This means that $\boldsymbol{U}_t \in \mathrm{M}(A \rtimes_\alpha G)$ and $\boldsymbol{U}_t^* = \boldsymbol{U}_{t^{-1}}$.

Ad (7). Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for A. Then for any $\mathfrak{F} \in \mathrm{C_c}(G, A)$ we have

$$
\boldsymbol{\pi}(e_\lambda)\mathfrak{F} \xrightarrow[\lambda\in\Lambda]{} \mathfrak{F}
$$

in the norm $\|\cdot\|_1$. Indeed, given $\varepsilon > 0$ we can first cover $\operatorname{supp}\mathfrak{F}$ with a finite collection of open sets $\mathcal{U}_1,\dots,\mathcal{U}_N$ with distinguished elements $t_k \in \mathcal{U}_k$ such that

$$\big\|\mathfrak{F}(t) - \mathfrak{F}(t_k)\big\| < \tfrac{\varepsilon}{3}$$

for all $t \in \mathcal{U}_k$ $(k = 1,\dots,N)$. Then we note that there is $\lambda_0 \in \Lambda$ such that for any $\lambda \geq \lambda_0$

$$\big\|e_\lambda\mathfrak{F}(t_k) - \mathfrak{F}(t_k)\big\| < \tfrac{\varepsilon}{3}$$

for $k = 1,\dots,N$. Therefore for any $t \in \mathcal{U}_k$ we have

$$\big\|e_\lambda\mathfrak{F}(t) - \mathfrak{F}(t)\big\| \leq \big\|e_\lambda\mathfrak{F}(t) - e_\lambda\mathfrak{F}(t_k)\big\| + \big\|e_\lambda\mathfrak{F}(t_k) - \mathfrak{F}(t_k)\big\| + \big\|\mathfrak{F}(t_k) - \mathfrak{F}(t)\big\| < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon$$

for all $\lambda \geq \lambda_0$. Since $\mathcal{U}_1,\dots,\mathcal{U}_N$ cover $\operatorname{supp}\mathfrak{F}$ we see that for $\lambda \geq \lambda_0$

$$\big\|\boldsymbol{\pi}(e_\lambda)\mathfrak{F} - \mathfrak{F}\big\|_1 < \varepsilon(\text{Haar measure of } \operatorname{supp}\mathfrak{F}).$$

Now we have already seen in the proof of (1) that $\|\cdot\| \leq \|\cdot\|_1$, so in fact

$$\boldsymbol{\pi}(e_\lambda)\mathfrak{F} \xrightarrow[\lambda\in\Lambda]{} \mathfrak{F}$$

in the C*-norm $\|\cdot\|$. In particular $\boldsymbol{\pi}(A)(A \rtimes_\alpha G)$ is dense in $A \rtimes_\alpha G$.

Ad (8). We shall first show that for any $\mathfrak{F} \in \mathrm{C_c}(G, A)$ we have

$$\boldsymbol{U}_t\mathfrak{F} \xrightarrow[t\to e]{} \mathfrak{F} \tag{5.5}$$

in the norm $\|\cdot\|_1$.

Let us then begin with noting that for any $\mathfrak{F} \in \mathrm{C_c}(G, A)$ and any $s, t \in G$ we have

$$\begin{aligned}\big\|\mathfrak{F}(s) - \alpha_t\big(\mathfrak{F}(t^{-1}s)\big)\big\| &\leq \big\|\mathfrak{F}(s) - \alpha_t\big(\mathfrak{F}(s)\big)\big\| + \big\|\alpha_t\big(\mathfrak{F}(s)\big) - \alpha_t\big(\mathfrak{F}(t^{-1}s)\big)\big\| \\ &= \big\|\mathfrak{F}(s) - \alpha_t\big(\mathfrak{F}(s)\big)\big\| + \big\|\mathfrak{F}(s) - \mathfrak{F}(t^{-1}s)\big\|.\end{aligned} \tag{5.6}$$

Fix $\varepsilon > 0$ and let $\mathcal{U}_1,\dots,\mathcal{U}_M$ be open subsets of G with distinguished elements $s_k \in \mathcal{U}_k$ $(k = 1,\dots,M)$ such that $\operatorname{supp}\mathfrak{F} \subset \mathcal{U}_1 \cup \dots \cup \mathcal{U}_M$ and for any k we have

$$\Big(s \in \mathcal{U}_k\Big) \implies \Big(\big\|\mathfrak{F}(s) - \mathfrak{F}(s_k)\big\| < \tfrac{\varepsilon}{6}\Big).$$

We have the following facts:

- since $\mathfrak{F}$ is uniformly continuous (see Proposition 3.3 in Subsection 3.2), there exists a neighborhood $\mathcal{V}_1$ of $e \in G$ such that
$$\Big(t \in \mathcal{V}_1\Big) \implies \Big(\big\|\mathfrak{F}(s) - \mathfrak{F}(t^{-1}s)\big\| < \tfrac{\varepsilon}{2}\Big).$$
for all $s \in G$,
- since α is continuous, there exists a neighborhood $\mathcal{V}_2$ of $e \in G$ such that
$$\Big(t \in \mathcal{V}_2\Big) \implies \Big(\big\|\mathfrak{F}(s_k) - \alpha_t\big(\mathfrak{F}(s_k)\big)\big\| < \tfrac{\varepsilon}{6}\Big).$$
for $k = 1,\dots,M$.

We put $\mathcal{V} = \mathcal{V}_1 \cap \mathcal{V}_2$.

Now for any k and any $s \in \mathcal{U}_k$ we have

$$\begin{aligned}\big\|\mathfrak{F}(s) - \alpha_t\big(\mathfrak{F}(s)\big)\big\| &\leq \big\|\mathfrak{F}(s) - \mathfrak{F}(s_k)\big\| + \big\|\mathfrak{F}(s_k) - \alpha_t\big(\mathfrak{F}(s_k)\big)\big\| + \big\|\alpha_t\big(\mathfrak{F}(s_k)\big) - \alpha_t\big(\mathfrak{F}(s)\big)\big\| \\ &= \big\|\mathfrak{F}(s) - \mathfrak{F}(s_k)\big\| + \big\|\mathfrak{F}(s_k) - \alpha_t\big(\mathfrak{F}(s_k)\big)\big\| + \big\|\mathfrak{F}(s_k) - \mathfrak{F}(s)\big\| < \tfrac{\varepsilon}{2}\end{aligned}$$

and

$$\big\|\mathfrak{F}(s) - \mathfrak{F}(t^{-1}s)\big\| < \tfrac{\varepsilon}{2}$$

provided $t \in \mathcal{V}$.

Thus if $t \in \mathcal{V}$ and $s \in G$ then s belongs to some $\mathcal{U}_k$ and we have by (5.6)

$$\big\|\mathfrak{F}(s) - \alpha_t\big(\mathfrak{F}(t^{-1}s)\big)\big\| < \varepsilon.$$

Consequently given $\varepsilon > 0$ there exists a neighborhood $\mathcal{V}$ of $e \in G$ such that for any $t \in \mathcal{V}$ we have

$$\big\|\mathfrak{F} - \boldsymbol{U}_t\mathfrak{F}\big\|_1 = \int\limits_G \big\|\mathfrak{F}(s) - \alpha_t\big(\mathfrak{F}(t^{-1}s)\big)\big\|\,dh^L(s) < \varepsilon(\text{Haar measure of } \operatorname{supp}\mathfrak{F})$$

which proves (5.5) for the norm $\|\cdot\|_1$.

Since $\|\cdot\| \leq \|\cdot\|_1$ we see that (5.5) holds for the C*-norm $\|\cdot\|$ on $\mathrm{C_c}(G,A) \subset A \rtimes_\alpha G$.

Let x be an arbitrary element of $A \rtimes_\alpha G$. Then for any $\varepsilon > 0$ there is $\mathfrak{F} \in \mathrm{C_c}(G,A)$ such that $\|x - \mathfrak{F}\| < \frac{\varepsilon}{3}$. Also there is a neighborhood $\mathcal{V}$ of $e \in G$ such that for $t \in \mathcal{V}$ we have $\big\|\mathfrak{F} - \boldsymbol{U}_t\mathfrak{F}\big\| < \frac{\varepsilon}{3}$. Therefore for $t \in \mathcal{V}$

$$\begin{aligned}\big\|x - \boldsymbol{U}_t x\big\| &\leq \big\|x - \mathfrak{F}\big\| + \big\|\mathfrak{F} - \boldsymbol{U}_t\mathfrak{F}\big\| + \big\|\boldsymbol{U}_t\mathfrak{F} - \boldsymbol{U}_t x\big\| \\ &= \big\|x - \mathfrak{F}\big\| + \big\|\mathfrak{F} - \boldsymbol{U}_t\mathfrak{F}\big\| + \big\|\mathfrak{F} - x\big\| < \varepsilon.\end{aligned}$$

This ends the proof of the fact that $\boldsymbol{U} \in \mathrm{Rep}(G, A \rtimes_\alpha G)$.

Ad (9). Take $t \in G$ and $a \in A$. For any $\mathfrak{F} \in \mathrm{C_c}(G,A)$ we have

$$\begin{aligned}\big(\boldsymbol{U}_t\boldsymbol{\pi}(a)\boldsymbol{U}_t^*\mathfrak{F}\big)(s) &= \alpha_t\Big(\big(\boldsymbol{\pi}(a)\boldsymbol{U}_t^*\mathfrak{F}\big)(t^{-1}s)\Big) \\ &= \alpha_t\big(a(\boldsymbol{U}_t^*\mathfrak{F})(t^{-1}s)\big) \\ &= \alpha_t\Big(a\alpha_{t^{-1}}\big(\mathfrak{F}(s)\big)\Big) \\ &= \alpha_t(a)\mathfrak{F}(s) = \big[\boldsymbol{\pi}\big(\alpha_t(a)\big)\mathfrak{F}\big](s)\end{aligned}$$

for any $s \in G$, so that $\boldsymbol{U}_t\boldsymbol{\pi}(a)\boldsymbol{U}_t^*\mathfrak{F} = \boldsymbol{\pi}\big(\alpha_t(a)\big)\mathfrak{F}$ for $\mathfrak{F}$ in a dense subset of $A \rtimes_\alpha G$. It follows that $(\boldsymbol{\pi}, \boldsymbol{U})$ is a covariant representation of (A, G, α) in $A \rtimes_\alpha G$.

Ad (10). We must show that for any covariant representation (π, U) of (A, G, α) in a C*-algebra B there exists a unique $\Phi \in \mathrm{Mor}(A \rtimes_\alpha G, B)$ such that (5.1) holds.

We shall first address existence of Φ. Lemma 5.6 gives us existence of the $*$-homomorphism $\pi \rtimes_\alpha U$ from $\mathrm{C_c}(G,A)$ into $\mathrm{M}(B)$. Now this homomorphism is norm decreasing by definition of the norm on $A \rtimes_\alpha G$. Therefore it extends to a $*$-homomorphism Φ from $A \rtimes_\alpha G$ to $\mathrm{M}(B)$. By Lemma 5.6 (1) we have $\Phi \in \mathrm{Mor}(A \rtimes_\alpha G, B)$.

Let us check that $\Phi(\boldsymbol{U}_t) = U_t$ for all $t \in G$. First note that if $\mathfrak{F} \in \mathrm{C_c}(G,A)$ then

$$\begin{aligned}\Phi(\boldsymbol{U}_t\mathfrak{F}) &= (\pi \rtimes_\alpha U)(\boldsymbol{U}_t\mathfrak{F}) \\ &= \int_G \pi\big((\boldsymbol{U}_t\mathfrak{F})(s)\big)U_s\,dh^L(s) \\ &= \int_G \pi\Big(\alpha_t\big(\mathfrak{F}(t^{-1}s)\big)\Big)U_s\,dh^L(s) \\ &= \int_G U_t\pi\big(\mathfrak{F}(t^{-1}s)\big)U_{t^{-1}s}\,dh^L(s) \\ &= U_t\int_G \pi\big(\mathfrak{F}(t^{-1}s)\big)U_{t^{-1}s}\,dh^L(s) \\ &= U_t\int_G \pi\big(\mathfrak{F}(s)\big)U_s\,dh^L(s) \\ &= U_t(\pi \rtimes_\alpha U)(\mathfrak{F}) = U_t\Phi(\mathfrak{F}).\end{aligned}$$

Therefore for any $\mathfrak{F} \in \mathrm{C_c}(G,A)$ and $b \in B$ we have

$$\Phi(\boldsymbol{U}_t)\Phi(\mathfrak{F})b = \Phi(\boldsymbol{U}_t\mathfrak{F})b = U_t\Phi(\mathfrak{F})b.$$

Since $\Phi\big(\mathrm{C_c}(G,A)\big)B$ is dense in B, we obtain $\Phi(\boldsymbol{U}_t) = U_t$.

Similarly if $a\in A$ and $\mathfrak{F}\in \mathrm{C_c}(G,A)$ then

$$\begin{aligned}\Phi\big(\boldsymbol{\pi}(a)\mathfrak{F}\big)&=(\pi\rtimes_\alpha U)\big(\boldsymbol{\pi}(a)\mathfrak{F}\big)\\&=\int_G \pi\Big(\big(\boldsymbol{\pi}(a)\mathfrak{F}\big)(s)\Big)U_s\,dh^L(s)\\&=\int_G \pi\big(a\mathfrak{F}(s)\big)U_s\,dh^L(s)\\&=\int_G \pi(a)\pi\big(\mathfrak{F}(s)\big)U_s\,dh^L(s)\\&=\pi(a)\int_G \pi\big(\mathfrak{F}(s)\big)U_s\,dh^L(s)\\&=\pi(a)(\pi\rtimes_\alpha U)(\mathfrak{F})=\pi(a)\Phi(\mathfrak{F})\end{aligned}$$

And we show that $\pi=\Phi\circ\boldsymbol{\pi}$ exactly as in the case of $\Phi(\boldsymbol{U}_t)=U_t$. We have proved that Φ defined as the extension of $\pi\rtimes_\alpha U$ defines an element of $\mathrm{Mor}(A\rtimes_\alpha G,B)$ such that (5.1) holds.

To prove the uniqueness of Φ let us first formulate one important lemma. The crucial ingredient is the fact that $(\boldsymbol{\pi},\boldsymbol{U})$ is a covariant representation of (A,G,α) in $A\rtimes_\alpha G$. Therefore, according to Lemma 5.6, there exists a map $\boldsymbol{\pi}\rtimes_\alpha\boldsymbol{U}$ defined on $\mathrm{C_c}(G,A)$ into $\mathrm{M}(A\rtimes_\alpha G)$.

Lemma 5.10. *The map $\boldsymbol{\pi}\rtimes_\alpha\boldsymbol{U}$ from $\mathrm{C_c}(G,A)$ to $\mathrm{M}(A\rtimes_\alpha G)$ extends to the identity map $A\rtimes_\alpha G\to A\rtimes_\alpha G$.*

Proof of Lemma 5.10. Let $\mathfrak{F}\in\mathrm{C_c}(G,A)$. Since $(\boldsymbol{\pi}\rtimes_\alpha\boldsymbol{U})(\mathfrak{F})$ is a multiplier of $A\rtimes_\alpha G$ we can compute $(\boldsymbol{\pi}\rtimes_\alpha\boldsymbol{U})(\mathfrak{F})\mathfrak{G}$ for $\mathfrak{G}\in\mathrm{C_c}(G,A)$:

$$\begin{aligned}\big[(\boldsymbol{\pi}\rtimes_\alpha\boldsymbol{U})(\mathfrak{F})\mathfrak{G}\big](t)&=\left[\int_G\boldsymbol{\pi}\big(\mathfrak{F}(s)\big)\boldsymbol{U}_s\,dh^L(s)\mathfrak{G}\right](t)\\&=\int_G\big[\boldsymbol{\pi}\big(\mathfrak{F}(s)\big)(\boldsymbol{U}_s\mathfrak{G})\big](t)\,dh^L(s)\\&=\int_G\mathfrak{F}(s)(\boldsymbol{U}_s\mathfrak{G})(t)\,dh^L(s)\\&=\int_G\mathfrak{F}(s)\alpha_s\big(\mathfrak{G}(s^{-1}t)\big)\,dh^L(s)=(\mathfrak{F}\star\mathfrak{G})(t).\end{aligned}$$

In other words $(\boldsymbol{\pi}\rtimes_\alpha\boldsymbol{U})(\mathfrak{F})\mathfrak{G}=\mathfrak{F}\star\mathfrak{G}$. Since this holds for all $\mathfrak{G}\in\mathrm{C_c}(G,A)$ and this space is dense in $A\rtimes_\alpha G$, we see that $(\boldsymbol{\pi}\rtimes_\alpha\boldsymbol{U})(\mathfrak{F})$ is the operator of left multiplication by $\mathfrak{F}$. This proves that $\boldsymbol{\pi}\rtimes_\alpha\boldsymbol{U}$ is the identity on $\mathrm{C_c}(G,A)$, so by continuity we get the desired result. ☐

It follows from Lemma 5.10 that

$$\left\{\int_G\boldsymbol{\pi}\big(\mathfrak{F}(s)\big)\boldsymbol{U}_s\,dh^L(s)\,\middle|\,\mathfrak{F}\in\mathrm{C_c}(G,A)\right\}=\mathrm{C_c}(G,A),\tag{5.7}$$

as subsets of $\mathrm{M}(A\rtimes_\alpha G)$ (in particular the left hand side is contained in $A\rtimes_\alpha G$).

Now let $\Phi_0\in\mathrm{Mor}(A\rtimes_\alpha G,B)$ be such that $\Phi_0\circ\boldsymbol{\pi}=\pi$ and $\Phi_0(\boldsymbol{U}_t)=U_t$ for all $t\in G$. Then we can apply Φ_0 to elements of the left hand side of (5.7):

$$\Phi_0\left(\int_G\boldsymbol{\pi}\big(\mathfrak{F}(s)\big)\boldsymbol{U}_s\,dh^L(s)\right)=\int_G\pi\big(\mathfrak{F}(s)\big)U_s\,dh^L(s)=(\pi\rtimes_\alpha U)(\mathfrak{F})=\Phi(\mathfrak{F})$$

so that Φ_0 coincides with Φ on a dense set. ☐

Let us also note the following corollary of Lemma (5.10):

Corollary 5.11. *Let $(A, G, , \alpha)$ be a* C*-*dynamical system and let $(\boldsymbol{\pi}, \boldsymbol{U})$ be the covariant representation of (A, G, α) in $A \rtimes_\alpha G$. Then the set*

$$\left\{ \int_G \boldsymbol{\pi}\big(\mathfrak{F}(s)\big)\boldsymbol{U}_s \, dh^L(s) \,\middle|\, \mathfrak{F} \in \mathrm{C_c}(G, A) \right\}$$

is a dense $*$-subalgebra of $A \rtimes_\alpha G$. Consequently the set

$$\left\{ \boldsymbol{\pi}(a) \int_G f(s)\boldsymbol{U}_s \, dh^L(s) \,\middle|\, f \in \mathrm{C_c}(G) \right\}$$

is a linearly dense subset of $A \rtimes_\alpha G$.

Let (A, G, α) be a C*-dynamical system. By definition of the crossed product any covariant representation (π, U) of (A, G, α) in a C*-algebra B there exists a unique $\Phi \in \mathrm{Mor}(A \rtimes_\alpha G, B)$ such that (5.1) holds. The next proposition states the (obvious) converse result.

Proposition 5.12. *Let (A, G, α) be a* C*-*dynamical system. Let B be a* C*-*algebra and let $\Phi \in \mathrm{Mor}(A \rtimes_\alpha G, B)$. Define π and U by (5.1). Then $\pi \in \mathrm{Mor}(A \rtimes_\alpha G, B)$, $U \in \mathrm{Rep}(G, B)$ and (π, U) is a covariant representation of (A, G, α) in B. Moreover the morphism form $A \rtimes_\alpha G$ to B associated to this covariant representation is Φ.*

5.3. Group C*-algebras.

Definition 5.13. Let G be a locally compact group and let α be the (necessarily) trivial action of G on the C*-algebra $\mathbb{C}$. The crossed product $\mathbb{C} \rtimes_\alpha G$ is called the *group* C*-*algebra* of G and is denoted by $\mathrm{C}^*(G)$. The associated representation $\boldsymbol{U} \in \mathrm{Rep}\big(G, \mathrm{C}^*(G)\big)$ is called the *universal representation* of G.

Corollary 5.14. *Let G be a locally compact group and let $\boldsymbol{U} \in \mathrm{Rep}\big(G, \mathrm{C}^*(G)\big)$ be the universal representation of G. Then for any* C*-*algebra B and any representation $V \in \mathrm{Rep}(G, B)$ there exists a unique $\Phi \in \mathrm{Mor}\big(\mathrm{C}^*(G), B\big)$ such that $V = \Phi \circ \boldsymbol{U}$.*

Remark 5.15. The group C*-algebra is determined uniquely up to isomorphism by the universal property described in Corollary 5.14.

Group C*-algebras are very important for general C*-algebra theory. However, their primary use comes from the theory of unitary representations of locally compact groups. If G is a locally compact group then representations of $\mathrm{C}^*(G)$ are in bijection with strongly continuous unitary representations of G. More precisely, by Proposition 4.23, for any Hilbert space H there is a bijection between strongly continuous unitary representations of G on H and elements of $\mathrm{Mor}\big(\mathrm{C}^*(G), \mathcal{K}(H)\big)$. This correspondence preserves direct sums, equivalence and quasi-equivalence. Any concept of the theory of representations of locally compact groups can be expressed in terms of group C*-algebras. Moreover many such concepts are easily generalized and fruitful in the theory of C*-algebras.

Let (A, G, α) be a C*-dynamical sysetm and let $(\boldsymbol{\pi}, \boldsymbol{U})$ be the universal covariant reresentation of (A, G, α) in $A \rtimes_\alpha G$. Then $\boldsymbol{U}$ is a representaion of G in $A \rtimes_\alpha G$, so it determines uniquely a morphism $\Phi_{\boldsymbol{U}} \in \mathrm{Mor}\big(\mathrm{C}^*(G), A \rtimes_\alpha G\big)$. Also $\boldsymbol{\pi} \in \mathrm{Mor}\big(A, A \rtimes_\alpha G\big)$. By Corollary 5.11 any morphism from $A \rtimes_\alpha G$ to a C*-algebra B is uniquely determined by its compositions with $\boldsymbol{\pi}$ nd $\Phi_{\boldsymbol{U}}$.

Moreover, it follows from the universal property of group C*-algebras that for any two locally compact groups G and G' and any continuous group homomorphism $\theta : G \to G'$ there is a unique $\Theta \in \mathrm{Mor}\big(\mathrm{C}^*(G), \mathrm{C}^*(G')\big)$ such that if $\boldsymbol{U}$ and $\boldsymbol{U}'$ are the universal representations of G and G' respectively then $\Theta(\boldsymbol{U}_t) = \boldsymbol{U}'_{\theta(t)}$ for all $t \in G$.

6. Examples

6.1. Transformation group C*-algebras.

A large class of important C*-algebras are the *transformation group* C*-*algebras,* i.e. crossed product C*-algebras arising from C*-dynamical systems of the type described in Example 4.27.

Let X be a locally compact space and let G be a locally compact group acting continuously on X. Then (4.11) defines an action $\alpha : G \ni t \mapsto \alpha_t \in \mathrm{Aut}\big(\mathrm{C}_\infty(X)\big)$ of G on $\mathrm{C}_\infty(X)$ and $\big(\mathrm{C}_\infty(X), G, \alpha\big)$ becomes a C*-dynamical system.

The crossed product $\mathrm{C}_\infty(X) \rtimes_\alpha G$ is called the *transformation group* C*-*algebra* of the system $\big(\mathrm{C}_\infty(X), G, \alpha\big)$.

We will describe a dense $*$-algebra of $\mathrm{C}_\infty(X) \rtimes_\alpha G$. Let us first note that

$$\mathrm{C_c}\big(G, \mathrm{C}_\infty(X)\big) \supset \mathrm{C_c}\big(G, \mathrm{C_c}(X)\big) \supset \mathrm{C_c}(G \times X)$$

and each subspace is dense in the larger one (an element $F \in \mathrm{C_c}(G \times X)$ gives rise to the function $G \ni t \mapsto F(t, \cdot)$ which is an element of $\mathrm{C_c}\big(G, \mathrm{C_c}(X)\big)$).

Now the formulas (5.2) give us product and involution on $\mathrm{C_c}(G \times X)$:

$$\begin{aligned}(F \star G)(t, x) &= \int_G F(s, x) G(s^{-1}t, s^{-1}x)\, dh^L(s),\\ F^*(t, x) &= \Delta(t)^{-1}\overline{F(t^{-1}, t^{-1}x)}\end{aligned}$$

(note that the involution $F \mapsto F^*$ is different from the standard involution on $\mathrm{C_c}(G \times X)$ coming from $\mathrm{C}_\infty(G \times X)$).

It follows from these formulas that $\mathrm{C_c}(G \times X)$ is indeed a $*$-subalgebra of $\mathrm{C}_\infty(X) \rtimes_\alpha G$. It is dense in $\mathrm{C_c}(G, A)$ in the norm $\|\cdot\|_1$, and so it is dense in the crossed product by Lemma 5.6 (2) (and the construction of the norm on $\mathrm{C}_\infty(X) \rtimes_\alpha G$).

Transformation group C*-algebras are objects encoding all information about the dynamical system (X, G). Moreover they are very useful in constructing examples of C*-algebras.

6.2. **Crossed product by a finite cyclic group.** In this subsection we let $G = \mathbb{Z}_N$. A C*-dynamical system (A, G, α) is given by specifying an automorphism of A (which we will also call α) whose N^{th} power is the identity automorphism. We shall describe the crossed product $A \rtimes_\alpha G$ completely.

Let us begin by noticing that if we map A into $M_N(A)$ via the injective $*$-homomorphism

$$A \ni a \longmapsto \pi(a) \begin{pmatrix} a &&&&& \\ & \alpha(a) &&&& \\ && \alpha^2(a) &&& \\ &&& \alpha^3(a) && \\ &&&& \ddots & \\ &&&&& \alpha^{N-1}(a) \end{pmatrix} \in M_N(A)$$

then the automorphism α will be implemented by conjugation with

$$U = \begin{pmatrix} 0 & I &&&&& \\ & 0 & I &&&& \\ && 0 & I &&& \\ &&& 0 & I && \\ &&&& \ddots & \ddots & \\ &&&&& 0 & I \\ I &&&&&& 0 \end{pmatrix}$$

Thus (π, U) is a covariant representation of (A, G, α) in $M_N(A)$.

Now let $\mathfrak{F} \in \mathrm{C_c}(G, A)$. Clearly $\mathfrak{F}$ is simply an ordered collection $(a_l)_{l=0}^{N-1}$ of elements of A.

It is easy to see that $(\pi\rtimes_\alpha U)(\mathfrak{F})$ is equal to the matrix

$$\begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_{N-3} & a_{N-2} & a_{N-1}\\
\alpha(a_{N-1}) & \alpha(a_0) & \alpha(a_1) & \cdots & \alpha(a_{N-4}) & \alpha(a_{N-3}) & \alpha(a_{N-2})\\
\alpha^2(a_{N-2}) & \alpha^2(a_{N-1}) & \alpha^2(a_0) & \cdots & \alpha^2(a_{N-5}) & \alpha^2(a_{N-4}) & \alpha^2(a_{N-3})\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\alpha^{N-3}(a_3) & \alpha^{N-3}(a_4) & \alpha^{N-3}(a_5) & \cdots & \alpha^{N-3}(a_0) & \alpha^{N-3}(a_1) & \alpha^{N-3}(a_2)\\
\alpha^{N-2}(a_2) & \alpha^{N-2}(a_3) & \alpha^{N-2}(a_4) & \cdots & \alpha^{N-2}(a_{N-1}) & \alpha^{N-2}(a_0) & \alpha^{N-2}(a_1)\\
\alpha^{N-1}(a_1) & \alpha^{N-1}(a_2) & \alpha^{N-1}(a_3) & \cdots & \alpha^{N-1}(a_{N-2}) & \alpha^{N-1}(a_{N-1}) & \alpha^{N-1}(a_0)
\end{pmatrix}.$$

Moreover $\pi\rtimes_\alpha U$ is injective. On the other hand the set of matrices of the form $(\pi\rtimes_\alpha U)(\mathfrak{F})$ is a $*$-subalgebra of $M_N(A)$ which is closed in norm of $M_N(A)$. Thus we see that $\mathrm{C_c}(G,A)$ carries a C*-norm in which it is already complete. Since the norm is unique on a C*-algebra (a standard fact proven using the notion of the *spectral radius*) there can be no other C*-norm on $\mathrm{C_c}(G,A)$ and so we have $\mathrm{C_c}(G,A)=A\rtimes_\alpha G$. Moreover $(\boldsymbol{\pi},\boldsymbol{U})=(\pi,U)$ is the universal representation of (A,G,α) in $A\rtimes_\alpha G$.

6.3. **Crossed product by a finite group acting on itself.** Let G be a group. Then there is always an action on G on itself by left multiplication. This action can be transferred to the space of functions defined on G. In particular, if G is a locally compact group then $G\ni t\mapsto\alpha_t\in\mathrm{Aut}\bigl(\mathrm{C}_\infty(G)\bigr)$ with $\bigl(\alpha_t(f)\bigr)(s)=f(t^{-1}s)$ is an action of G on $\mathrm{C}_\infty(G)$. In this subsection we shall describe the C*-algebra $\mathrm{C}_\infty(G)\rtimes_\alpha G$ for a finite group G. In what follows we shall use the symbol $\mathrm{Fun}(G)$ to denote the algebra of all functions from G to $\mathbb{C}$ (it coincides with $\mathrm{C}(G)$, $\mathrm{C}_\infty(G)$ and $\mathrm{C_c}(G)$ and saves the reader a lot of confusion).

Let π be the representation of $\mathrm{Fun}(G)$ in $L^2(G)$ as multiplication operators: for $f\in\mathrm{Fun}(G)$ and $\psi\in L^2(G)$

$$\bigl(\pi(f)\psi\bigr)(t)=f(t)\psi(t)$$

for all $t\in G$. Also let U be the left regular representation of G in $L^2(G)$: for $\psi\in L^2(G)$ and $s\in G$

$$(U_s\psi)(t)=\psi(s^{-1}t)$$

for all $t\in G$. First let us note that for any $g\in\mathrm{Fun}(G)$ the operator

$$\int\limits_G g(s)U_s\,dh^L(s)=\sum_{s\in G}g(s)U_s$$

is the operator of convolution with the function g: for any $\psi\in L^2(G)$ we have

$$\left(\sum_{s\in G}g(s)U_s\psi\right)(t)=\sum_{s\in G}g(s)\psi(s^{-1}t)=(g\star\psi)(t)$$

for all $t\in G$.

Recall that in case of a general C*-dynamical system (A,G,α) the $*$-algebra $\mathrm{C_c}(G,A)$ is dense in $A\rtimes_\alpha G$. Now in our case $\mathrm{C_c}(G,A)$ is the set $\mathrm{Fun}\bigl(G,\mathrm{Fun}(G)\bigr)$ of all functions from G to $\mathrm{Fun}(G)$. Therefore it is a finite dimensional algebra. Since finite dimensional spaces are complete in all norms, and norm on a C*-algebra is unique, it is enough to exhibit one C*-norm on $\mathrm{Fun}\bigl(G,\mathrm{Fun}(G)\bigr)$ and this must be the norm coming from the construction of the crossed product. The easiest candidate is the norm inherited from $\mathrm{B}\bigl(L^2(G)\bigr)$.

In conclusion we find that for a finite group G the crossed product $\mathrm{C}_\infty(G)\rtimes_\alpha G$ is isomorphic to the C*-algebra of operators on $L^2(G)$ of the form

$$\sum_{s\in G}\pi\bigl(\mathfrak{F}(s)\bigr)U_s,$$

where $\mathfrak{F}\in\mathrm{Fun}\bigl(G,\mathrm{Fun}(G)\bigr)$ (cf. Corollary 5.11). We shall now prove that this C*-algebra is isomorphic to M_N, where N is the order of G.

Quite obviously $\mathrm{Fun}\big(G,\mathrm{Fun}(G)\big)$ is isomorphic as a vector space to the algebra $\mathrm{Fun}(G\times G)$ of all functions on $G\times G$. The product and involution on this algebra is given by

$$(F\star F')(t,s)=\sum_{u\in G}F(u,s)F'(u^{-1}t,u^{-1}s),$$
$$F^*(t,s)=\overline{F(t^{-1},t^{-1}s)}$$

for all $F,F'\in\mathrm{Fun}(G\times G)$ and all $t,s\in G$ (cf. Subsection 6.1).

Let us label the elements of G:

$$G=\big\{s_1,\dots,s_N\big\}$$

Now to any function $F\in\mathrm{Fun}(G\times G)$ we can assign a matrix $m(F)$ by setting

$$\big[m(F)\big]_{k,l}=F(s_ks_l^{-1},s_k)$$

for $k,l=1,\dots,N$.

It is easy to see that the map $F\mapsto m(F)$ sets up a $*$-isomorphism of algebra $\mathrm{Fun}(G\times G)$ with the algebra M_N. Indeed, let $F,G\in\mathrm{Fun}(G\times G)$. Then

$$\begin{aligned}\big[m(F)m(G)\big]_{k,l}&=\sum_{r=1}^{N}\big[m(F)\big]_{kr}\big[m(G)\big]_{rl}\\&=\sum_{r=1}^{N}F(s_ks_r^{-1},s_k)G(s_rs_l^{-1},s_r)\\&=\sum_{u\in G}F(u,s_k)G(u^{-1}s_ks_l^{-1},u^{-1}s_k)\\&=(F\star G)(s_ks_l^{-1},s_k)=\big[m(F\star G)\big]_{kl}.\end{aligned}$$

Similarly

$$\big[m(F^*)\big]_{kl}=F^*(s_ks_l^{-1},s_k)=\overline{F(s_ls_k^{-1},s_l)}==\overline{\big[m(F)\big]_{lk}}=\big(m(F)^*\big)_{kl}.$$

As the map $F\mapsto m(F)$ is clearly surjective and $\dim\mathrm{Fun}(G\times G)=N^2$, it follows that we have $\mathrm{C}_\infty(G)\rtimes_\alpha G\cong M_N$.

Remark 6.1. It is a fact that for any locally compact group G the crossed product of $\mathrm{C}_\infty(G)$ by the action α of left translation by elements of G is always isomorphic to the algebra $\mathcal{K}\big(L^2(G)\big)$:

$$\mathrm{C}_\infty(G)\rtimes_\alpha G\cong\mathcal{K}\big(L^2(G)\big).$$

6.4. **C*-algebra of a semidirect product of groups.** In this subsection we shall present a construction which gave birth to the notion of a crossed product.

Let N and P be locally compact groups and let $\alpha:P\ni p\mapsto\alpha_p\in\mathrm{Aut}(N)$ be a homomorphism such that for each $n\in N$ the map $P\ni p\mapsto\alpha_p(n)\in N$ is continuous. Then the locally compact space $N\times P$ can be made into a locally compact group with the multiplication

$$(n,p)(m,q)=(n\alpha_p(m),pq).$$

The neutral element for this multiplication is, of course (e_N,e_P) and the inverse operation is given by

$$(n,p)^{-1}=\big(\alpha_{p^{-1}}(n^{-1}),p^{-1}\big).$$

The group so obtained is called the *semidirect product* of N by the action α of P and is denoted by $N\rtimes_\alpha P$. The groups N and P can be identified with closed subgroups of $N\rtimes_\alpha P$ via

$$\begin{aligned}\imath_N:&N\ni n\longmapsto(n,e_P),\\\imath_P:&P\ni p\longmapsto(e_N,p).\end{aligned}$$

It is easy to see that $\imath_N(N)$ is a normal subgroup of $N\rtimes_\alpha P$ and that we have

$$\imath_P(p)\imath_N(n)\imath_P(p)^{-1}=\imath_N\big(\alpha_p(n)\big).$$

Recall that with any (continuous) group homomorphism there is an associated morphism between the corresponding group C*-algebras. Let us denote the morphisms arising from $\imath_N$ and $\imath_P$ by the same symbols:

$$\imath_N \in \mathrm{Mor}\big(\mathrm{C}^*(N), \mathrm{C}^*(N \rtimes_\alpha P)\big), \qquad \imath_P \in \mathrm{Mor}\big(\mathrm{C}^*(P), \mathrm{C}^*(N \rtimes_\alpha P)\big).$$

Also, for each $p \in P$ the automorphism α_p of N induces an automorphism of $\mathrm{C}^*(N)$ (which we shall denote by the same symbol). This way we get a C*-dynamical system $\big(\mathrm{C}^*(N), P, \alpha\big)$.

Proposition 6.2. *We have* $\mathrm{C}^*(N \rtimes_\alpha P) = \mathrm{C}^*(N) \rtimes_\alpha P$.

Proof. We will show that $\mathrm{C}^*(N) \rtimes_\alpha P$ is isomorphic to $\mathrm{C}^*(N \rtimes_\alpha P)$ by exhibiting a representation $\boldsymbol{U}^{N \rtimes_\alpha P}$ of $N \rtimes_\alpha P$ in $\mathrm{C}^*(N) \rtimes_\alpha P$ and proving that it has the universal property of the group C*-algebra of $N \rtimes_\alpha P$ (cf. Corollary 5.14).

Let us first introduce some notation. The universal representations of N and P in $\mathrm{C}^*(N)$ and $\mathrm{C}^*(P)$ will be denoted respectively by $\boldsymbol{U}^N$ and $\boldsymbol{U}^P$. Further let $(\boldsymbol{\pi}, \boldsymbol{U})$ be the universal covariant representation of the C*-dynamical system $\big(\mathrm{C}^*(N), P, \alpha\big)$ in $\mathrm{C}^*(N) \rtimes_\alpha P$.

For $(n,p) \in N \rtimes_\alpha P$ let

$$\boldsymbol{U}^{N \rtimes_\alpha P}_{(n,p)} = \boldsymbol{\pi}(\boldsymbol{U}^N_n)\boldsymbol{U}_p. \tag{6.1}$$

We have $\boldsymbol{U}^{N \rtimes_\alpha P} \in \mathrm{Rep}\big(N \rtimes_\alpha P, \mathrm{C}^*(N) \rtimes_\alpha P\big)$.

Now let B be a C*-algebra and let $U \in \mathrm{Rep}(N \rtimes_\alpha P, B)$. We have representations

$$N \ni n \longmapsto U_{\imath_N(n)} \in \mathrm{M}(B),$$
$$P \ni p \longmapsto U_{\imath_P(p)} \in \mathrm{M}(B)$$

which give rise to unique morphisms $\pi^N \in \mathrm{Mor}\big(\mathrm{C}^*(N), B\big)$ and $\pi^P \in \mathrm{Mor}\big(\mathrm{C}^*(P), B\big)$ such that

$$\begin{aligned} \pi^N(\boldsymbol{U}^N_n) &= U_{\imath_N(n)}, \\ \pi^P(\boldsymbol{U}^P_p) &= U_{\imath_P(p)} \end{aligned} \tag{6.2}$$

for all $n \in N$ and $p \in P$. Let us also denote the restriction of U to P by U^P: $U^P_p = U_{\imath_P(p)}$ for all $p \in P$.

Now we note that (π^N, U^P) is a covariant representation of $\big(\mathrm{C}^*(N), P, \alpha\big)$ in B. Therefore there exists a unique $\Phi \in \mathrm{Mor}\big(\mathrm{C}^*(N) \rtimes_\alpha P, B\big)$ such that

$$\begin{aligned} \Phi \circ \boldsymbol{\pi} &= \pi^N, \\ \Phi(\boldsymbol{U}_p) &= U^P_p \end{aligned} \tag{6.3}$$

for all $p \in P$.

We have by (6.1), (6.3), (6.2) and the definition of U^P:

$$\Phi(\boldsymbol{U}^{N \rtimes_\alpha P}_{(n,p)}) = \Phi\big(\boldsymbol{\pi}(\boldsymbol{U}^N_n)\big)\Phi(\boldsymbol{U}_p) = \pi^N(\boldsymbol{U}^N_n)U^P_p = U_{\imath_N(n)}U_{\imath_P(p)} = U_{(n,p)}$$

for all $(n,p) \in N \rtimes_\alpha P$. Clearly this property is equivalent to (6.3), so that Φ with this property is also unique.

We have thus shown that the pair $\big(\mathrm{C}^*(N) \rtimes_\alpha P, \boldsymbol{U}^{N \rtimes_\alpha P}\big)$ has the universal property of the croup C*-algebra of $N \rtimes_\alpha P$. ☐

6.5. $\mathrm{C}^*(\mathbb{Z}_2 * \mathbb{Z}_2)$**.** The C*-algebra in the title of this subsection is very interesting and it is an example of three different constructions presented so far. Namely it is a group C*-algebra, but it can also be viewed as a crossed product which is, in fact, a transformation group C*-algebra. There is yet anther interpretation for this C*-algebra. It can be shown to be the universal C*-algebra generated by two projections. However, we shall not address this point here.

Let us begin by showing that $\mathbb{Z}_2 * \mathbb{Z}_2$ is isomorphic to $\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2$, where the period two automorphism α of $\mathbb{Z}$ is given by $\alpha(n) = -n$ (of course, it is the only non trivial such automorphism).

The group $\mathbb{Z}_2 * \mathbb{Z}_2$ consists of the following elements

$$\big\{\mathcal{E}, x, y, xy, yx, xyx, yxy, xyxy, yxyx, \ldots\big\}$$

where by $\mathcal{E}$ we denoted the empty word (neutral element of the free product) and x and y are the non trivial elements of the two copies of $\mathbb{Z}_2$ sitting inside $\mathbb{Z}_2 * \mathbb{Z}_2$. Clearly $\{\mathcal{E}, x\}$ is a subgroup isomorphic to $\mathbb{Z}_2$, while

$$\{\mathcal{E}, xy, yx, xyxy, yxyx, \dots\}$$

(words with even number of letters) is a normal subgroup isomorphic to $\mathbb{Z}$. The isomorphism is given by $xy \mapsto 1$. It is trivial to see that conjugation by x implements the automorphism α. All this shows that we indeed have $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes_\alpha \mathbb{Z}_2$. In particular, by the results of Subsection 6.4, we have

$$\mathrm{C}^*(\mathbb{Z}_2 * \mathbb{Z}_2) \cong \mathrm{C}^*(\mathbb{Z} \rtimes_\alpha \mathbb{Z}_2) \cong \mathrm{C}^*(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}_2$$

To procede with the analysis of $\mathrm{C}^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ we must make use of the result identifying the group C*-algebra of an Abelian group with the C*-algebra of continuous functions vanishing at infinity on the dual group (Subsection 7.2). In our case this means that

$$\mathrm{C}^*(\mathbb{Z}) \cong \mathrm{C}(\mathbb{T}).$$

Moreover the automorphism α becomes the map

$$\mathrm{C}(\mathbb{T}) \ni f \longmapsto \alpha(f) \in \mathrm{C}(\mathbb{T}),$$

where $\alpha(f)\big(e^{2\pi\mathrm{i}t}\big) = f\big(e^{2\pi\mathrm{i}(1-t)}\big)$ for all $t \in [0,1]$. We have so far reached the conclusion that

$$\mathrm{C}^*(\mathbb{Z}_2 * \mathbb{Z}_2) \cong \mathrm{C}(\mathbb{T}) \rtimes_\alpha \mathbb{Z}_2.$$

Now the C*-algebra on the right hand side is a crossed product by an action of a finite cyclic group which we analyzed in Subsection 6.2.

We know that $\mathrm{C}(\mathbb{T}) \rtimes_\alpha \mathbb{Z}_2$ is isomorphic to the subalgebra of $M_2\big(\mathrm{C}(\mathbb{T})\big)$ consisting of elements

$$\left\{ \begin{pmatrix} f & g \\ \alpha(g) & \alpha(f) \end{pmatrix} \middle| \, f, g \in \mathrm{C}(\mathbb{T}) \right\}.$$

We obtain an isomorphic C*-algebra if we conjugate these matrices by

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

which yields

$$\left\{ \begin{pmatrix} \big(f + \alpha(f)\big) - \big(g + \alpha(g)\big) & \big(f - \alpha(f)\big) + \big(g - \alpha(g)\big) \\ \big(f - \alpha(f)\big) - \big(g - \alpha(g)\big) & \big(f + \alpha(f)\big) + \big(g + \alpha(g)\big) \end{pmatrix} \middle| \, f, g \in \mathrm{C}(\mathbb{T}) \right\}. \tag{6.4}$$

Note that the off diagonal entries vanish at the points $\{1, -1\} \subset \mathbb{T}$. Moreover the diagonal entries of the matrix are invariant and the off-diagonal change sign under α. This means that it is enough to consider $e^{2\pi i t}$ for $t \in [0, \frac{1}{2}]$ (each element of (6.4) is determined by its values on half of the circle $\mathbb{T}$). All this shows that $\mathrm{C}(\mathbb{T}) \rtimes_\alpha \mathbb{Z}_2$ is ismorphic to the C*-algebra of M_2 valued continuous functions on an interval such whose values at the endpoints are diagonal:

$$\left\{ \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \middle| \, f_{11}, f_{12}, f_{21}, f_{22} \in \mathrm{C}([0,1]),\ f_{12}(0) = f_{21}(0) = f_{12}(1) = f_{21}(1) = 0 \right\}.$$

6.6. **Reduced crossed products.** Let (A, G, α) be a C*-dynamical system. Let π_0 be a faithful and non degenerate representation of A on a Hilbert space $\mathcal{H}$. As in Proposition 5.7 we can construct a covariant representation (π, U) of (A, G, α) in $\mathcal{K}(H)$, where $H = \mathcal{H} \otimes L^2(G)$. Proposition 5.7 (4) asserts that in this case $\pi \rtimes_\alpha U$ is injective. Let $\Lambda \in \mathrm{Mor}\big(A \rtimes_\alpha G, \mathcal{K}(H)\big)$ be the corresponding morphism (extension of $\pi \rtimes_\alpha U$ from $\mathrm{C_c}(G, A)$ to the whole of $A \rtimes_\alpha G$). One might ask two important questions:

- Does the C*-algebra of $\Lambda(A \rtimes_\alpha G) \subset \mathrm{B}(H)$ depend on the choice of π_0?
- Is Λ an injective map?

The answer to the first question is quite satisfactory: the C*-algebra $\Lambda(A \rtimes_\alpha G) \subset B(H)$ does not depend on the representation π_0 of A. The proof (which falls beyond the scope of these notes) may be found in [3, Theorem 7.7.5].

The answer to the second question is more complicated. There is no general statement, but we have many partial results. The most important is that if G is *amenable* then Λ is injective and so the image of Λ is isomorphic to $A \rtimes_\alpha G$. Amenability is a property of (topological) groups which will be explained in more detail in Subsection 7.5. We shall, however, refrain from developing the necessary harmonic analysis to prove any of the statements concerning amenability. There is ample literature on this subject and the book [1] is strongly suggested as one of the best expositions of the theory.

All Abelian groups are amenable and so are all compact (in particular finite) groups. Any closed subgroup of an amenable groups is amenable. Moreover if a group has a amenable normal subgroup such that the quotient is amenable then the original group is amenable. Solvable groups are amenable.

On the other hand semisimple Lie groups are not amenable (unless they are compact). The free group on two generators is not amenable. It is important to note that amenability is not a purely group theoretical concept. It depends on the topology of the considered group. For example $SO(3)$ is a compact group (hence amenable), but it contains the free group on two generators which fails to be amenable. Therefore $SO(3)$ with discrete topology is not amenable.

The injectivity of Λ depends also on the action α of G on A. It can happen that a non amenable group acts in such a way that Λ is an isomorphism.

Definition 6.3. Let (A, G, α) be a C*-dynamical system. The image of $A \rtimes_\alpha G$ under the map Λ corresponding to any faithful representation π_0 of A is called the *reduced crossed product* of A by the action α of G. It is denoted by $A \rtimes_{\alpha,r} G$.

In case of $A = \mathbb{C}$ we obtain the image of $C^*(G)$ under the left regular representation in $L^2(G)$. This C*-algebra is called the *reduced group* C*-*algebra* of G and is denoted by $C^*_r(G)$.

7. Advanced theory

7.1. **Pontriagin duality for Abelian locally compact groups.** Let G be an Abelian topological group. A *character* of G is a continuous homomorphism $G \to \mathbb{T}$. It is easy to see that the set $\widehat{G}$ of all characters of G is a group under pointwise multiplication. Moreover the topology on G makes it possible to make $\widehat{G}$ into a toplogical group.

The topology on $\widehat{G}$ is the well known *compact-open* topology. In this topology a base of neighborhoods of $x \in \widehat{G}$ is given by the collection $\big(\mathcal{U}_{K,U}\big)$ where K is a compact subset of G and U is an open subset of $\mathbb{T}$ and
$$\mathcal{U}_{K,U} = \big\{y \in \widehat{G} \big| y(K) \subset U\big\}.$$
Since $\mathbb{T}$ is a metric space, there is a well defined notion of uniform convergence of functions with values in $\mathbb{T}$. The compact-open topology on $\widehat{G}$ is precisely the topology of uniform convergence on compact subsets of G.

Definition 7.1. Let G be an Abelian topological group. The group $\widehat{G}$ of all characters of G with compact-open topology is called the *Pontiagin dual* of G. The value of $x \in \widehat{G}$ on $t \in G$ is denoted by $\langle x, t\rangle$ and the map
$$\widehat{G} \times G \ni (x,t) \longmapsto \langle x, t\rangle \in \mathbb{T}$$
is called the *pairing* between G and $\widehat{G}$.

The next theorem is called the *duality theorem.*

Theorem 7.2. *Let G be an Abelian locally compact group and let $\widehat{G}$ be the Pontriagin dual of G. Then*

1. *$\widehat{G}$ is a locally compact Abelian group,*
2. *for any $t \in G$ the function $\widehat{G} \ni x \mapsto \langle x, t\rangle \in \mathbb{T}$ is a character of $\widehat{G}$,*
3. *the map $G \ni t \mapsto \langle \cdot, t\rangle \in \widehat{\widehat{G}}$ is an isomorphism of topological groups.*

Here are some of the important facts concerning duality of locally compact Abelian groups:

- The dual of $\mathbb{R}$ is $\mathbb{R}$, the dual of $\mathbb{Z}$ is $\mathbb{T}$ and the dual of $\mathbb{Z}_N$ is $\mathbb{Z}_N$.
- The dual of a compact group is discrete (and hence the dual of a discrete group is compact).
- The dual of a topological vector space $\mathfrak{X}$ can be identified with $\mathfrak{X}^*$ via
$$\mathfrak{X}^*\times\mathfrak{X}\ni(\varphi,x)\longmapsto\langle\varphi,x\rangle=e^{\mathrm{i}\varphi(x)}\in\mathbb{T}.$$
This shows that without the assumption of local compactness the duality theorem fails.
- The passage from G to $\widehat{G}$ is a functor: given a continuous homomorphism of locally compact Abelian groups $\theta:G\to G'$ there is a continuous homomorphism $\widehat{\theta}:\widehat{G'}\to\widehat{G}$ determined by
$$\langle z,\theta t\rangle=\bigl\langle\widehat{\theta}(z),t\bigr\rangle$$
for all $z\in\widehat{G'}$, $t\in G$ (in other words $\widehat{\theta}(z)=z\circ\theta$). We have $\widehat{\mathrm{id}_G}=\mathrm{id}_{\widehat{G}}$ for any G and $\widehat{\theta_1\circ\theta_2}=\widehat{\theta}_2\circ\widehat{\theta}_1$ whenever θ_1 and θ_2 can be composed.
- The duality functor commutes with finite Cartesian products. (This follows from the fact that in the category of Abelian groups we have both the operation of Cartesian product and the operation of direct sum and they coincide.) In case of an infinite product or sum, direct products are transformed into direct sums and vice versa.
- If G is an Abelian topological group then so is $\boldsymbol{d}G$ – the group G with discrete topology. The dual $\widehat{\boldsymbol{d}G}$ of $\boldsymbol{d}G$ is a compact Abelian group containing $\widehat{G}$ (it is the group of *all* homomorphisms $G\to\mathbb{T}$, not just the continuous ones). Applying this procedure to $\widehat{G}$ instead of G we obtain a compact abelian group $\widehat{\boldsymbol{d}\widehat{G}}$ containing G. This compact group is called the *Bohr compactification* of G and is denoted by $\boldsymbol{b}G$. It is relevant for the theory of almost periodic functions.

7.2. C*-**algebra of an Abelian locally compact group.** In this section we shall state a theorem which gives a very convenient description of $\mathrm{C}^*(G)$ for an Abelian locally compact group G.

Let G be an Abelian locally compact group and let $\widehat{G}$ be the dual group of G. The duality between G and $\widehat{G}$ is described by the pairing
$$\widehat{G}\times G\ni(x,t)\longmapsto\langle x,t\rangle.$$
Let $\boldsymbol{U}$ be the universal representation of G in $\mathrm{C}^*(G)$. We also have a representation $\boldsymbol{V}\in\mathrm{Rep}\bigl(G,\mathrm{C}_\infty(\widehat{G})\bigr)$ given by
$$(\boldsymbol{V}_tf)(x)=\langle x,t\rangle f(x)$$
for all $f\in\mathrm{C}_\infty(\widehat{G})$, $x\in\widehat{G}$, $t\in G$. By the universal property of $\mathrm{C}^*(G)$ there is a unique $\Phi\in\mathrm{Mor}\bigl(\mathrm{C}^*(G),\mathrm{C}_\infty(\widehat{G})\bigr)$ such that for all $t\in G$
$$\Phi(\boldsymbol{U}_t)=\boldsymbol{V}_t.$$
Let us take an element f of $\mathrm{C}^*(G)$ from the dense $*$-subalgebra $\mathrm{C_c}(G)$ identified with
$$\left\{\int\limits_G f(s)\boldsymbol{U}_s\,dh(s)\;\middle|\;f\in\mathrm{C_c}(G)\right\}$$
(cf. Corollary 5.11). Then $\Phi(f)$ is a bounded continuous function on $\widehat{G}$. For $x\in\widehat{G}$ we have
$$\bigl(\Phi(f)\bigr)(x)=\int\limits_G f(s)\langle x,s\rangle\,dh(s). \tag{7.1}$$
The reader will have recognized in (7.1) the classical Fourier transform.

Theorem 7.3. *The map* (7.1) *extends to an isomorphism of* $\mathrm{C}^*(G)$ *onto* $\mathrm{C}_\infty\bigl(\widehat{G}\bigr)$.

Sketch of proof. Probably the best way to approach this problem is to use Gelfand's theory of commutative C*-algebras (see [4, Chapter 4]). First we identify $\widehat{G}$ with the *spectrum* of the commutative C*-algebra $C^*(G)$, i.e. the set $\mathrm{Mor}\big(C^*(G),\mathbb{C}\big)$. This identification is practically obvious given the connection between representations of G and those of $C^*(G)$. It is also a simple matter to see that the compact-open topology on $\widehat{G}$ coincides with the weak* topology $\mathrm{Mor}\big(C^*(G),\mathbb{C}\big)$ inherits from the dual space of $C^*(G)$.

Next we note that (7.1) is nothing else, but the *Gelfand transform* applied to an element $f\in C_c(G)$ seen as a subset of $C^*(G)$. It is known that the Gelfand transform is an isomorphism for commutative C*-algebras and its image is the algebra of all continuous functions vanishing at infinity on the spectrum. ☐

7.3. **Landstad's theorem.** In this subsection we want to state the theorem of M. Landstad which gives a precise description of those C*-algebras B for which there exists a C*-dynamical system (A,G,α) with Abelian G such that $B\cong A\rtimes_\alpha G$. It turns out a C*-algebra B is a crossed product of some C*-algebra by an action of a commutative group G if and only if B has the structure of a *G-product.*

Definition 7.4. Let G be an Abelian locally compact group and let B be a C*-algebra. We say that B is a *G-product* if there exist

- $U\in\mathrm{Rep}(G,B)$,
- $\widehat{\alpha}\in\mathrm{Act}\big(\widehat{G},B\big)$

such that for any $t\in G$ and $x\in\widehat{G}$ we have

$$\widehat{\alpha}_x(U_t)=\langle x,t\rangle U_t.$$

Definition 7.5. Let G be an Abelian locally compact group and let B be a G-product with $U\in\mathrm{Rep}(G,B)$ and $\widehat{\alpha}\in\mathrm{Act}\big(\widehat{G},B\big)$ as in Definition 7.4. An element $b\in\mathrm{M}(B)$ satisfies *Landstad's conditions* if

(1) $\widehat{\alpha}_x(b)=b$ for all $x\in\widehat{G}$,
(2) $b\int_G f(s)U_s\,dh(s)\in B$ for all $f\in C_c(G)$,
(3) the map $G\ni t\mapsto U_tbU_t^*\in\mathrm{M}(B)$ is continuous (in the norm topology).

Remark 7.6. In literature (e.g. [3, Section 7.8.2]) condition (2) is written as

$$b\int\limits_G f(s)U_s\,dh(s),\int\limits_G f(s)U_s\,dh(s)\,b\in B,$$

but the combination of (1) and (3) can be used to show that once $b\int_G f(s)U_s\,dh(s)$ is an element of B, the product $\int_G f(s)U_s\,dh(s)\,b$ must also lie in B.

Theorem 7.7 (Landstad)**.** *Let G be an Abelian locally compact group and let B be a G-product with $U\in\mathrm{Rep}(G,B)$ and $\widehat{\alpha}\in\mathrm{Act}\big(\widehat{G},B\big)$ as in Definition 7.4. Let A be the set of all elements of $\mathrm{M}(B)$ satisfying Landstad's conditions. Then*

(1) *A is a C*-algebra,*
(2) *for any $t\in G$ the map $\alpha_t:A\ni a\mapsto U_taU_t^*\in A$ is an automorphism of A and the mapping $\alpha:G\ni t\mapsto\alpha_t$ is an action of G on A,*
(3) *$B\cong A\rtimes_\alpha G$,*
(4) *any C*-dynamical system (C,G,β) such that $B\cong C\rtimes_\beta G$ is covariantly isomorphic to (A,G,α).*

Let us see that, conversely, for any C*-dynamical system (A,G,α) with Abelian G the crossed product $A\rtimes_\alpha G$ carries a natural structure of a G-product. Let $(\boldsymbol{\pi},\boldsymbol{U})$ be the universal covariant representation of (A,G,α) in $A\rtimes_\alpha G$. Then we already have a representation of G in $A\rtimes_\alpha G$, namely $\boldsymbol{U}$.

For each $x \in \widehat{G}$ let $\boldsymbol{U}^x$ be the representation of G in $A \rtimes_\alpha G$ given by

$$\boldsymbol{U}^x_t = \langle x, t\rangle \boldsymbol{U}_t$$

for all $t \in G$. Then $(\boldsymbol{\pi}, \boldsymbol{U}^x)$ is a covariant representation of (A, G, α) in $A \rtimes_\alpha G$, and so there exists a unique $\widehat{\alpha}_x \in \mathrm{Mor}(A \rtimes_\alpha G, A \rtimes_\alpha G)$ such that

$$\begin{aligned} \widehat{\alpha}_x \circ \boldsymbol{\pi} &= \boldsymbol{\pi}, \\ \widehat{\alpha}_x(\boldsymbol{U}_t) &= \boldsymbol{U}^x_t = \langle x, t\rangle \boldsymbol{U}_t. \end{aligned} \tag{7.2}$$

It is easy to see (by considering $\widehat{\alpha}_{x^{-1}}$) that $\widehat{\alpha}_x$ is an automorphism of $A \rtimes_\alpha G$ for each $x \in \widehat{G}$. To see that $\widehat{\alpha} : \widehat{G} \ni x \mapsto \widehat{\alpha}_x$ is an action of $\widehat{G}$ on $A \rtimes_\alpha G$ we must check that for each $c \in A \rtimes_\alpha G$ the map $x \mapsto \widehat{\alpha}_x(c)$ is continuous. For this it is enough to consider first c from the dense $*$-subalgebra $\mathrm{C_c}(G, A)$, i.e. c of the form

$$c = \int_G \boldsymbol{\pi}\big(\mathfrak{F}(s)\big)\boldsymbol{U}_s \, dh(s)$$

(cf. Lemma 5.10 and Corollary 5.11). Then $\widehat{\alpha}_x(c) \in \mathrm{C_c}(G, A)$ and

$$\begin{aligned} \big\|c - \widehat{\alpha}_x(c)\big\|_1 &= \left\| \int_G \boldsymbol{\pi}\big(\mathfrak{F}(s)\big)\boldsymbol{U}_s \, dh(s) - \int_G \boldsymbol{\pi}\big(\mathfrak{F}(s)\big)\langle x, s\rangle \boldsymbol{U}_s \, dh(s) \right\|_1 \\ &= \left\| \int_G \boldsymbol{\pi}\big(\mathfrak{F}(s)\big)\big(1 - \langle x, s\rangle\big)\boldsymbol{U}_s \, dh(s) \right\|_1 \\ &\leq \sup_{s \in \operatorname{supp} \mathfrak{F}} \big|1 - \langle x, s\rangle\big| \, \|c\|_1 \end{aligned}$$

Now recall from Subsection 7.1 that the topology on $\widehat{G}$ is precisely such that the above quantity goes to 0 as $x \to e \in \widehat{G}$. In other words

$$\big\|c - \widehat{\alpha}_x(c)\big\|_1 \xrightarrow[x \to e \in \widehat{G}]{} 0.$$

Since the C*-norm $\|\cdot\|$ on $A \rtimes_\alpha G$ is smaller than $\|\cdot\|_1$, we see that also

$$\big\|c - \widehat{\alpha}_x(c)\big\| \xrightarrow[x \to e \in \widehat{G}]{} 0.$$

Now all we need to do is use the standard $\frac{\varepsilon}{3}$ trick (as e.g. in the proof of Theorem 5.9 (7)) to get continuity of $x \mapsto \widehat{\alpha}_x(c)$ for any $c \in A \rtimes_\alpha G$.

We have thus constructed an action $\widehat{\alpha}$ of $\widehat{G}$ on $A \rtimes_\alpha G$. Together with $\boldsymbol{U} \in \mathrm{Rep}(G, A \rtimes_\alpha G)$ it endows $A \rtimes_\alpha G$ with structure of a G-product (by the second line of (7.2)).

Definition 7.8. The action $\widehat{\alpha}$ of $\widehat{G}$ on $A \rtimes_\alpha G$ constructed above is called the *dual of* α or simply the *dual action.*

It is not so hard to see that Landstad's conditions applied to the canonical structure of a G-product on $A \rtimes_\alpha G$ coming from the dual action select precisely the image $\boldsymbol{\pi}(A)$ of A in $\mathrm{M}(A \rtimes_\alpha G)$.

7.4. **Takai's duality for crossed products.** Let G be an Abelian locally compact group and let α be an action of G on a C*-algebra A. We have already seen in the final paragraphs of Subsection 7.3 that there is an action of $\widehat{G}$ on $A \rtimes_\alpha G$. It is interesting to see what happens if we take the crossed product by this action.

In the formulation of the next theorem we shall use the right regular representation ρ of G on $L^2(G)$. It is given by

$$(\rho_t \psi)(s) = \Delta(s)^{\frac{1}{2}} \psi(st)$$

for all $s, t \in G$ and $\psi \in L^2(G)$. Here Δ is the modular function of G. Of course, for Abelian G we have $\Delta \equiv 1$.

Theorem 7.9 (Takai)**.** *Let (A, G, α) be a* C*-*dynamical system with G Abelian. Let $\widehat{\alpha}$ be the dual action of $\widehat{G}$ on $A \rtimes_\alpha G$.*

(1) *the* C^**-algebra* $(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$ *is isomorphic to* $A \otimes \mathcal{K}\big(L^2(G)\big)$,
(2) *the action* $\widehat{\widehat{\alpha}}$ *dual to* $\widehat{\alpha}$ *on* $A \otimes \mathcal{K}\big(L^2(G)\big)$ *is* $\alpha \otimes \mathrm{Ad}_\rho$, *where* ρ *is the right regular representation of* G *on* $L^2(G)$, *i.e. for any* $a \in A$, *and* $c \in \mathcal{K}\big(L^2(G)\big)$ *we have*
$$\widehat{\widehat{\alpha}}_t(a \otimes c) = \alpha_t(a) \otimes \rho_t c \rho_t^*$$
for all $t \in G$.

Taking in Theorem 7.9 $A = \mathbb{C}$ we obtain the generalization of the result given in Subsection 6.2 to all locally compact Abelian groups. The proof of Takai's theorem is quite complicated and all details can be found in [3, Section 7.9]. A corresponding result for crossed products of von Neumann algebras was proved a few years before Takai's paper by M. Takesaki.

7.5. **Amenable groups.** In this section we will try to shed some light on the concept of amenability of locally compact groups (which we brushed on in Subsection 6.6). A very good reference on this subject is the book [1].

Definition 7.10. Let G be a group and let $\mathcal{F}$ be a vector space of functions $G \to \mathbb{C}$ containing the constant functions and invariant under left translations, i.e. if $f \in \mathcal{F}$ then for any $t \in G$ the function
$$f_t : G \ni s \longmapsto f(t^{-1}s) \in \mathbb{C}$$
also belongs to $\mathcal{F}$. A linear functional m on $\mathcal{F}$ is called *left invariant mean* on $\mathcal{F}$ if
- $m(f_t) = m(f)$ for all $t \in G$,
- $m(f) \geq 0$ if f is a non negative function,
- $m(I) = 1$, where I is the constant function equal to 1.

Of course, there is a corresponding notion of a right invariant mean.

Definition 7.11.
(1) A group G is called *amenable* if there is a left invariant mean on $\ell^\infty(G)$.
(2) A locally compact group G is called *amenable* if there is a left invariant mean on $\mathrm{C_b}(G)$.

It is somewhat misleading that the two notions defined above are referred to by the same word. There are simple examples of groups which are amenable as locally compact groups, but are not amenable as groups without topology.

We have already given a short list of Amenable groups in Subsection 6.6. Let us now prove that the free group on two generators $\mathbb{F}_2$ is not amenable.

To see this note that if m is a left invariant mean on a discrete group G then the map
$$\mu : 2^G \ni \Omega \longmapsto m(\chi_\Omega)$$
is a finitely additive, left invariant measure defined on 2^G (the power set of G). Note that we have $\mu(G) = 1$.

Now let a and b be generators of $\mathbb{F}_2$. Elements of $\mathbb{F}_2$ are words on the two letters a and b and we can always write such words in a *reduced form.* This means that each $x \in \mathbb{F}_2$ can be written as a finite product
$$x = a^{p_1} b^{p_2} a^{p_3} \cdots$$
where either all indices $p_2, p_3, \ldots$ are different from 0 or they are all equal to 0 (and in this case $x = a^{p_1}$). For $n \in \mathbb{Z}$ let $S(n)$ be the subset of $\mathbb{F}_2$ sonsisting of reduced words beginning with a^n. In particular $S(0)$ is set of words beginning with a non zero power of b and the empty word. For different $n, m \in \mathbb{Z}$ the sets $S(n)$ and $S(m)$ are disjoint.

We have the following facts:
- the left translation $x \mapsto ax$ maps $S(n)$ onto $S(n+1)$ for all $n \in \mathbb{Z}$,
- the left translation $x \mapsto bx$ maps $S(n)$ with $n \neq 0$ into $S(0)$.

The first fact implies that the measure μ must assign to each $S(n)$ the same number. On the other hand all the sets $S(n)$ are disjoint and they are all contained in $\mathbb{F}_2$ whose measure is 1. Thus
$$\mu\big(S(n)\big) = 0 \tag{7.3}$$

forall $n \in \mathbb{Z}$.

Now the second fact above says that $\mu\bigl(S(0)\bigr)$ is greater or equal to

$$\mu\left(\bigcup_{n\neq 0} S(n)\right)$$

because the set $\bigcup_{n\neq 0} S(n)$ is mapped onto a subset of $S(0)$ by the map $x \mapsto bx$. In view of (7.3), this means that

$$\mu\left(\bigcup_{n\neq 0} S(n)\right) = 0.$$

On the other hand we have

$$\bigcup_{n\neq 0} S(n) \cup S(0) = \mathbb{F}_2$$

so

$$\mu\left(\bigcup_{n\neq 0} S(n)\right) + \mu\bigl(S(0)\bigr) = 1$$

which is a contradiction.

Now it is amusing to note that the two matrices

$$\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$$

generate a free subgroup in any of the matrix groups that contain them (e.g. $SL(2,\mathbb{Z})$ or $SL(2,\mathbb{C})$).

The next theorem gives a profound interpretation of amenability of locally compact groups. The proof may be found in [3, Section 7.3].

Theorem 7.12. *Let G be a locally compact group and let $\lambda \in \mathrm{Mor}\bigl(\mathrm{C}^*(G), \mathrm{C}^*_r(G)\bigr)$ be the morphism associated to the left regular representation of G on $L^2(G)$. Then G is amenable if and only if λ is an isomorphism.*

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