# Summary + keywords

- **Topological manifolds:**

  Open sets, neighbourhood, homeomorphism, subspace topology, manifold with boundary.

- **Smooth manifolds:**

  Transition functions, chart ≡ local coordinate, atlas, diffeomorphism $F : M \to N$, immersion, embedding, m-submanifold property.

- **Tangent space:**

  $T_p(M)$, basis of $T_p(M)$ : $\left\{ \varphi_*^{-1} \left( \frac{\partial}{\partial x^j} \Big|_{\varphi(p)} \right) \right\}_{j=1}^{m}$, coordinate frame, $F : M \to N \Rightarrow \exists\, F^*$ and $F_*$, curve on $M \Rightarrow$ tangent vector.

- <u>Vector fields :</u>

$X : M \ni p \mapsto X_p \in T_p(M)$ , one vector in each tangent space, smooth vector fields $\mathfrak{X}(M)$, $C^\infty$-module + Lie algebra structure.

- <u>Flow :</u>

Integral curves and local flow $\Leftrightarrow X \in \mathfrak{X}(M)$, singular, regular point for $X$, Lie derivative along $X$.

- <u>Tensors :</u>

Multilinear maps from $\overbrace{V \times \dots V}^{r \text{ times}} \times \overbrace{V^* \times \dots V^*}^{s \text{ times}} \longrightarrow \mathbb{R}$

Permutation group, $\mathcal{S}, \mathcal{A}$ acting on $\mathcal{T}^r(V)$

tensor product $\otimes$ and wedge product $\wedge$,

$\mathcal{T}(V)$, $\Sigma(V)$, $\Lambda(V)$ (of dimension $2^{\dim(V)}$).

exterior algebra

tensor algebra

- Dual basis :

coordinate coframe

Existence of dual basis, $\{(dx^j)_p\}_{j=1}^m$ basis of $T_p^*(M)$ satisfying $(dx^j)_p(E_{i,p}) = \delta_{ij}$,

For $f \in C^\infty(M)$, $(df)_p \in T_p^*(M)$, $df$ = differential of $f$.

- Tensor fields :

Function from $M \ni p$ to tensors on $T_p(M)$, smooth tensor fields, symmetric / antisymmetric tensor fields. Special cases :

1) $\mathcal{T}_1^0(M) \equiv \mathfrak{X}(M) = \{$vector fields$\}$

2) $\mathcal{T}_0^1(M) = \{$ covector fields $\}$

3) $\Sigma^2(M) = \{$ symmetric field of bilinear maps on $T_p(M)$ $\}$

Riemannian manifold = $\exists\, \phi \in \Sigma^2(M)$ positive definite.