Proving $\mathbb{R}^n$ is a Topological Manifold

HADIKO Rifqi Aufa Sholih (062101868)

Special Mathematics Lecture: Differential Geometry (Fall 2024)

This report aims to show that $\mathbb{R}^n$ is an n-dimensional topological manifold while recalling all necessary definitions and solve some of the corresponding exercises taken from the lecture note [2].

First, lets start with the definition of topological manifold to make our direction clear.

Definition 1 (Topological manifold)**.** *A m-dimensional* topological manifold *consists in a topological space* $(M, \mathcal{T})$ *which is Hausdorff and second countable, and such that for any* $p \in M$ *there exist* $U \in \mathcal{N}_p$ *homeomorphic to an open subset of* $\mathbb{R}^m$*.*

To prove $\mathbb{R}^n$ is an n-dimensional topological manifold, we have to show that $\mathbb{R}^n$ is Hausdorff and second countable, and show there exist homeomorphism between an open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^n$.

$\mathbb{R}^n$ with the usual topology

We begin with the definition of topological space. Then, we will examine the usual topology on $\mathbb{R}^n$ through the corresponding exercise.

Definition 2 (Topological space)**.** *A topological space* $(M, \mathcal{T})$ *is a set* M *together with a family* $\mathcal{T}$ *of subsets of* M *satisfying*

(i) $\emptyset$ *and* M *belong to* $\mathcal{T}$*,*

(ii) If $V_\alpha \in \mathcal{T}$*, then* $\cup_\alpha V_\alpha \in \mathcal{T}$*, (*$\mathcal{T}$ *stable for arbitrary unions)*

(iii) If $V_i \in \mathcal{T}$*, then* $\cap_{i=1}^N V_i \in \mathcal{T}$*, (*$\mathcal{T}$ *stable for finite intersections).*

The elements of $\mathcal{T}$ *are usually called* the open sets *of* M*, and one says that* $\mathcal{T}$ *defines* the topology *of* M*.*

Definition 3 (Ball). *A* ball *on* $\mathbb{R}^n$ *is defined as* $B(x,r) := \{p \in \mathbb{R}^n \mid d(p,x) < r\}$ *with* $d(p,q) = \left[\sum_{i=1}^{n} (p^i - q^i)^2\right]^{1/2}$ *is the* distance *between two points* p *and* q *in* $\mathbb{R}^n$.

Exercise 1.1.3 ($\mathbb{R}^n$ and the usual topology). *Recall that a set* U *on* $\mathbb{R}^n$ *is open if for any* $x \in U$ *there exists a ball* $B(x,r)$ *centred at* x *and of radius* $r > 0$ *such that* $B(x,r) \subset U$. *With this definition, check that the set of all open sets on* $\mathbb{R}^n$ *defines a topology on* $\mathbb{R}^n$. *This topology corresponds to the usual topology on* $\mathbb{R}^n$*, and justifies why the open sets can be called open, according to Definition 1.1.1. Note also that we simply write* $\mathbb{R}^n$ *for the topological space, without mentioning explicitly the set of open sets.*

Proof: Set $\mathcal{T}$ as a family of all open sets on $\mathbb{R}^n$. We will show that $(\mathbb{R}^n, \mathcal{T})$ is a topological space by proving the following properties.

(*i*) $\emptyset$ and $\mathbb{R}^n$ belong to $\mathcal{T}$.
Since $\emptyset$ does not contains any element, then the only ball contained in $\emptyset$ is an empty set, so $\emptyset \in \mathcal{T}$. Also, we have $\mathbb{R}^n \in \mathcal{T}$ since there always exist $B(x,r)$ for any $x \in \mathbb{R}^n$.

(*ii*) If $V_\alpha \in \mathcal{T}$, then $\cup_\alpha V_\alpha \in \mathcal{T}$.
Let $\forall \alpha \in \mathbb{N}$, U_α is an open set on $\mathbb{R}^n$. Since for any $y \in U_\alpha$ there exist a ball $B(y,r) \subset U_\alpha$, then for any $x \in \cup_\alpha U_\alpha$ there exist a ball $x \in B(x,r) \subset \cup_\alpha U_\alpha$. Therefore, arbitrary unions of the open sets is also an open set. Hence, we obtain for $U_\alpha \in \mathcal{T}$, then $\cup_\alpha U_\alpha \in \mathcal{T}$.

(*iii*) If $V_i \in \mathcal{T}$, then $\cap_{i=1}^N V_i \in \mathcal{T}$.
Let $\forall i \in \mathbb{N}$, V_i is an open set on $\mathbb{R}^n$. Since for any $y \in V_i$ there exist a ball $B(y,r) \subset V_i$, then for any $x \in \cap_{i=1}^N V_i$ there exist a ball $x \in B(x,r) \subset \cap_{i=1}^N V_i$. Therefore, finite unions of the open sets is also an open set. Hence, we obtain for $V_i \in \mathcal{T}$, then $\cap_{i=1}^N V_i \in \mathcal{T}$.

Hence, $(\mathbb{R}^n, \mathcal{T})$ is a topological space. By the definition of topological space, the elements of $\mathcal{T}$ is called the open sets of $\mathbb{R}^n$, which justifies the open sets to be called open. □

From now on, we will simply write $\mathbb{R}^n$ for the topological space with the usual topology.

Hausdorff property of $\mathbb{R}^n$

Let us recall the definition of neighbourhood and Hausdorff property. Then, we will proceed to proof Hausdorff property of $\mathbb{R}^n$.

Definition 4 (Neighbourhood)**.** *Let $(M, \mathcal{T})$ be a topological space, and let $p \in M$. A neighbourhood of p is any element $V \in \mathcal{T}$ satisfying $p \in V$. The set of all neighbourhoods of p is denoted by $\mathcal{N}_p$.*

Definition 5 (Hausdorff property)**.** *A topological space $(M, \mathcal{T})$ is* Hausdorff *if for any p_1, $p_2 \in M$ with $p_1 \neq p_2$ there exist $V_1 \in \mathcal{N}_{p_1}$ and $V_2 \in \mathcal{N}_{p_2}$ such that $V_1 \cap V_2 \in \emptyset$.*

Exercise 1.1.6 (partly) Show that $\mathbb{R}^n$ is Hausdorff.

Proof: Recall that by definition, a ball is defined as $B(x, r) = \{p \in \mathbb{R}^n \mid d(p, x) < r\}$ with $d(p, q) = \left[\sum_{i=1}^{n} (p^i - q^i)^2\right]^{1/2}$ is the distance between two points p and q in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, $B(x, r)$ is an open set on $\mathbb{R}^n$, then $B(x, r) \in \mathcal{N}_x$.

Let us consider two point p_1 and p_2 on $\mathbb{R}^n$ such that $d(p_1, p_2) = 2\epsilon$ with $\epsilon > 0$. Then, for $B(p_1, \epsilon) \in \mathcal{N}_{p_1}$ and $B(p_2, \epsilon) \in \mathcal{N}_{p_2}$, we have $B(p_1, \epsilon) \cap B(p_2, \epsilon) = \emptyset$ because the midpoint $m \in \mathbb{R}^n$ between p_1 and p_2 with $d(m, p_1) = d(m, p_2) = \epsilon$ is not included in both set. Therefore, $\mathbb{R}^n$ is Hausdorff. $\square$

$\mathbb{R}^n$ is second countable

We will define basis and second countable, and use the definition, lemma, and proposition taken from [1], page 323, to prove that $\mathbb{R}^n$ is second countable.

Definition 6 (Basis)**.** *A subset $\mathcal{B} := \{V_\alpha\} \subset \mathcal{T}$ is a* basis *for the topological space $(M, \mathcal{T})$ if for any $p \in M$ and any $U \in \mathcal{N}_p$, there exist $V \in \mathcal{B}$ such that $p \in V \subset U$.*

Definition 7 (Second countable)**.** *A topological space $(M, \mathcal{T})$ is called* second countable *if it possesses a basis which is countable (meaning that its elements can be indexed by $\mathbb{N}$).*

Definition 8 (Rational point)**.** *A point in $\mathbb{R}^n$ is* rational *if all of its coordinates are rational numbers.*

Lemma 1. *Every open set in $\mathbb{R}^n$ contains a rational point.*

Proof: An open set U in $\mathbb{R}^n$ contains an open ball $B(p, r)$ by the definition of open set. Further, let us evaluate the biggest open cube $\prod_{i=1}^{n} I_i$ centred on p that can fit inside the open ball $B(p, r)$, where I_i is open interval centred on p^i, i.e. $(p^i - s, p^i + s)$ with $s \in \mathbb{R}$.

Observe that if one has the maximal sized cube inside a ball, then the corners of the cube touches the boundary of the ball. Then, we can define a corner point $c \in \mathbb{R}^n$ for well oriented cube as a point that is equidistant from its center point p with the distance $d(c,p) = r$ and $|c^i - p^i| = s$ for all $i \in \{1, ..., n\}$. Using the definition of distance in $\mathbb{R}^n$, one has

$$\begin{aligned} d(c,p) &= \left[\sum_{i=1}^{n} (c^i - q^i)^2\right]^{1/2} \\ &= \left[\sum_{i=1}^{n} s^2\right]^{1/2} \\ &= (ns^2)^{1/2} = r \\ \implies s &= \frac{r}{\sqrt{n}} \end{aligned}$$

Hence, the open interval is given by $I_i = (p^i - r/\sqrt{n}, p^i + r/\sqrt{n})$.

From real analysis, it is known that every open interval in $\mathbb{R}$ contains a rational number. Then, we can choose $q^i \in I_i$ such that $q^i \in \mathbb{Q}$. Therefore, for $\prod_{i=1}^{n} I_i \subset B(p,r) \subset U$, rational point q with $q^i \in \mathbb{Q}$ for all $i \in \{1, ..., n\}$ exist in every open set. $\square$

Proposition 1. *The collection $\mathcal{B}_{rat}$ of all open balls in $\mathbb{R}^n$ with rational centers and rational radii is a basis for $\mathbb{R}^n$.*

Proof: Let us consider an open set U in $\mathbb{R}^n$ and a point $p \in U$, there exist an open ball $B(p, r')$ with $r' > 0$ such that $p \in B(p, r') \subset U$. From the result in real analysis, we can choose $r \in (0, r')$ such that $r \in \mathbb{Q}$, then we have $p \in B(p,r) \subset U$. Consider the smaller ball $B(p, r/2)$, using the Lemma 1, there exist a rational point $q \in B(p, r/2)$ since the ball is an open set.

From $q \in B(p, r/2)$, we have $d(p,q) < r/2$, so that $p \in B(q, r/2)$. For $x \in B(q, r/2)$, $d(x,q) < r/2$. By using the triangle inequality, we obtain

$$d(x,p) \leq d(x,q) + d(p,q) < \frac{r}{2} + \frac{r}{2} = r \implies d(x,p) < r \implies x \in B(p,r).$$

Since x is any element of $B(q, r/2)$, then $B(q, r/2) \subset B(p,r)$. This relation further implies $B(q, r/2) \subset B(p,r) \subset U$.

Hence, there exist $B(q, r/2) \in \mathcal{B}_{rat}$ such that $p \in B(q, r/2) \subset U$, which means $\mathcal{B}_{rat}$ form a basis. $\square$

Exercise 1.1.9 ($\mathbb{R}^n$ is second countable). *Check that $\mathbb{R}^n$ with the usual topology is a second countable topological space.*

Proof: From Proposition 1, $\mathcal{B}_{rat}$ form a basis for $\mathbb{R}^n$. Observe that the centers of the balls in $\mathcal{B}_{rat}$ are indexed by $\mathbb{Q}^n$ and the radii are indexed by $\mathbb{Q}^+$, where both $\mathbb{Q}^n$ and $\mathbb{Q}^+$ are countable set, which implies $\mathcal{B}_{rat}$ is countable basis. Since $\mathcal{B}_{rat}$ is a countable basis for $\mathbb{R}^n$, then $\mathbb{R}^n$ is second countable. □

Homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$

Here we will define continuous map and homeomorphism. Homeomorphism we will use to show that $\mathbb{R}^n$ is a topological manifold is the most trivial map.

Definition 9 (Continuous map). *Let $(M, \mathcal{T})$ and $(N, \mathcal{S})$ be topological spaces, and let $f : M \to N$. The map f is* continuous *if for any $U \in \mathcal{S}$, its preimage $f^{-1}(U)$ is an element of $\mathcal{T}$, or more precisely for any $U \in \mathcal{S}$, one has $f^{-1}(U) \in \mathcal{T}$, where $f^{-1}(U) := \{x \in M \mid f(x) \in U\}$.*

Definition 10 (Homeomorphism). *Let $(M, \mathcal{T})$ and $(N, \mathcal{S})$ be topological spaces, and let $f : M \to N$ be a bijective map. If f and its inverse f^{-1} are continuous, then f is called a* homeomorphism*.*

Let us consider an identity map $id : \mathbb{R}^n \to \mathbb{R}^n$ such that for $x \in \mathbb{R}^n$ one has $id(x) = x$. Its inverse map is equivalent to itself since $id^{-1}(id(x)) = x \implies id^{-1}(x) = x \implies id^{-1} = id$. Then, since for any open set U on $\mathbb{R}^n$, its preimage $id^{-1}(U) = id(U) = U$ is also an open set on $\mathbb{R}^n$, the identity map id is continuous.

Obviously, the identity map id is a bijective map. Since id and its inverse ($id^{-1} = id$) is continuous, then id is a homeomorphism.

$\mathbb{R}^n$ is an n-dimensional topological manifold

We have proved that $\mathbb{R}^n$ with the usual topology is Hausdorff and second countable, and such that for any $p \in \mathbb{R}^n$ there exist an open set $U \in \mathcal{N}_p$ homeomorphic to an open subset of $\mathbb{R}^n$ ($id(U) \in \mathbb{R}^n$). Therefore, according to Definition 1, $\mathbb{R}^n$ is an n-dimensional topological manifold.

References

[1] L. Tu, *An Introduction to Manifolds*, Second Edition, Universitext, Springer, New York, 2011.

[2] S. Richard, *Special Mathematics Lecture: Differential Geometry*. 2024.