Homomorphisms between Manifolds

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Using **Definition 1 .3.4** o f ([1], p g. 9), f or a ny s mooth m ap $F : M \to N$ b etween t wo smooth manifolds and for any $p \in M$, one sets

$$F^* : C^\infty(F(p)) \to C^\infty(p) \quad \text{with} \quad F^*(f) := f \circ F, \quad \forall f \in C^\infty(F(p))$$

and

$$F_* : T_p(M) \to T_{F(p)}(N) \quad \text{with} \quad F_*(X_p)(f) := X_p(F^*(f)), \quad \forall X_p \in T_p(M) \text{ and } f \in C^\infty(F(p)).$$

Theorem 2.1.2.

In the framework introduced above, the map F^* is a homomorphism of algebras while the map F_* is a homomorphism of vector spaces. In addition, if $H = G \circ F$ is the composition of two smooth maps between manifolds, then $H^* = F^* \circ G^*$ and $H_* = G_* \circ F_*$.

Exercise 2.1.3.

Prove **Theorem 2.1.2**.

1. Proof that F^* is a Homomorphism of Algebras

Let $f, g \in C^\infty(F(p))$. In order to prove the map F^* is a homomorphism of algebras, we need to prove the linearity and the multiplicative property. Namely, for any scalar $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned} F^*(\alpha f + \beta g) &= (\alpha f + \beta g) \circ F \\ &= \alpha f \circ F + \beta g \circ F \\ &= \alpha F^*(f) + \beta F^*(g), \end{aligned}$$

Moreover, one also has

$$\begin{aligned} F^*(fg) &= (fg \circ F) \\ &= (f \circ F)(g \circ F) \\ &= F^*(f)F^*(g). \end{aligned}$$

Thus, the F^* preserves linearity and multiplication, making it a homomorphism of algebras.

2. Proof that F_* is a Homomorphism of Vector Spaces

To check F_* is a homomorphism of vector spaces, we only need to check the linearity property. Let $X_p, Y_p \in T_p(M)$ and scalars $\alpha, \beta \in \mathbb{R}$. For any $f \in C^\infty(F(p))$, we have the following

$$F_*(\alpha X_p + \beta Y_p)(f) = (\alpha X_p + \beta Y_p)(F^*(f)) = \alpha X_p(F^*(f)) + \beta Y_p(F^*(f)) = \alpha F_*(X_p)(f) + \beta F_*(Y_p)(f).$$

Thus, F_* is linear, and therefore a homomorphism of vector spaces.

3. Proof of Composition Rules

Let $H = G \circ F : M \to P$ be the composition of two smooth maps $F : M \to N$ and $G : N \to P$.

Pullback Composition: $H^* = F^* \circ G^*$

For any $h \in C^\infty(H(p))$, we have:

$$H^*(h) = h \circ H = h \circ (G \circ F) = (h \circ G) \circ F = F^*(G^*(h)) = (F^* \circ G^*)(h).$$

Since this holds for all $h \in C^\infty(H(p))$, we conclude that $H^* = F^* \circ G^*$.

Pushforward Composition: $H_* = G_* \circ F_*$

Let $X_p \in T_p(M)$. Then $H_*(X_p) \in T_{H(p)}(P)$ is defined by its action on smooth functions $h \in C^\infty(H(p))$:

$$H_*(X_p)(h) := X_p(H^*(h)).$$

Using the composition $H^* = F^* \circ G^*$ we proved above, we find:

$$H_*(X_p)(h) = X_p((F^* \circ G^*)(h)) = F_*(X_p)(G^*(h)).$$

Now, by definition of G_*, this equals

$$G_*(F_*(X_p))(h) = (G_* \circ F_*)(X_p)(h).$$

Since this holds for all $h \in C^\infty(H(p))$, we conclude that $H_* = G_* \circ F_*$.

Remark

The map $F_* : T_p(M) \to T_{F(p)}(N)$ is often called the differential of F, and various notations are used for it: dF, DF or F'. Here and in the sequel, we shall use $*$ as a subscript when it goes in the direction of F, while we shall use $*$ as the superscript when it goes in the direction opposite to F, as in $F^* : C^\infty(F(p)) \to C^\infty(p)$.

In the special case of a diffeomorphism $F : M \to N$, as introduced in **Definition 1.3.5**, the previous statement can be strengthened. An isomorphism is a bijective map preserving the structures, with its inverse also preserving the structures.

Theorem 2.1.4.

Let $F : M \to N$ be a diffeomorphism, and let $p \in M$. Then $F_* : T_p(M) \to T_{F(p)}(N)$ is an isomorphism.

Proof:
Since F is a diffeomorphism, it is both bijective and smooth, and its inverse $G = F^{-1}$ is also smooth. Therefore, we have that $G \circ F = \mathrm{id}_M$ and $F \circ G = \mathrm{id}_N$, where id_M and id_N are the identity maps on M and N, respectively.

By **Theorem 2.1.2**, we know that if $H = G \circ F$, then $H_* = G_* \circ F_*$ and $H^* = F^* \circ G^*$. Applying this to the case where G is the inverse of F, we have

$$G_* \circ F_* = \mathrm{id}_{T_p(M)} \quad \text{and} \quad F_* \circ G_* = \mathrm{id}_{T_{F(p)}(N)}.$$

This means that F_* has a two-sided inverse, which is G_* and this implies that F_* is bijective.

Moreover, since F_* is a homomorphism of vector spaces (by **Theorem 2.1.2**), it preserves the linear structure. Hence, from the bijectivity of F_*, $F_* : T_p(M) \to T_{F(p)}(N)$ is a linear isomorphism.

References

[1] Serge Richard. *Special Mathematics Lecture: Differential Geometry*. 2024.