SML Differential geometry (Fall 2024)
Exercise 3.1.1. (p.25)

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Let V be a finite dimensional real vector space, and V^* be its dual space, namely the set of all linear maps from V to $\mathbb{R}$.

Exercise 0.1. [1, Exercise 3.1.1.] Prove the following statements:

1. V^* is itself a vector space.

2. If the dimension of V is n, then the dimension of V^* is also n.

Proof. 1. Let v_i^*, v_j^*, and v_k^* be arbitrary elements of V^*, and α and β be in $\mathbb{R}$. We define

(a) $(v_i^* + v_j^*)(v) := v_i^*(v) + v_j^*(v) \quad$ for all $v \in V$,

(b) $(\alpha v_i^*)(v) := \alpha\left(v_i^*(v)\right)$.

Then $v_i^* + v_j^*$ and αv_i^* are in V^*. Indeed, for all v_1, v_2 in V^*, and for all α, β in $\mathbb{R}$,

$$\begin{aligned}
(v_1^* + v_2^*)(v_1 + \alpha v_2) &= v_1^*(v_1 + \alpha v_2) + v_2^*(v_1 + \alpha v_2) \\
&= v_1^*(v_1) + \alpha v_1^*(v_2) + v_2^*(v_1) + \alpha v_2^*(v_2) \\
&= (v_1^* + v_2^*)(v_1) + \alpha(v_1^* + v_2^*)(v_2). \\
(\alpha v_1^*)(v_1 + \beta v_2) &= \alpha\left(v_1^*(v_1) + \beta v_1^*(v_2)\right) \\
&= (\alpha v_1^*)(v_1) + \beta(\alpha v_1^*)(v_2).
\end{aligned}$$

Under this addition and scalar multiplication, we can check the following properties for all v_i^*, v_j^*, and v_k^* in V^*:

(a) (Associativity of addition)

$$\begin{aligned}
\left(v_i^* + (v_j^* + v_k^*)\right)(v) &= \left(v_i^*(v) + \left(v_j^*(v) + v_k^*(v)\right)\right) \\
&= \left(\left(v_i^*(v) + v_j^*(v)\right) + v_k^*(v)\right) \\
&= \left(\left(v_i^* + v_j^*\right) + v_k^*\right)(v) \quad \text{for all } v \in V.
\end{aligned}$$

(b) (Commutivity of addition)

$$\begin{aligned}
\left(v_i^* + v_j^*\right)(v) &= \left(v_i^*(v) + v_j^*(v)\right) \\
&= \left(v_j^*(v) + v_i^*(v)\right) \\
&= \left(v_j^* + v_i^*\right)(v) \quad \text{for all } v \in V.
\end{aligned}$$

(c) (Existence of a zero)
We define an element 0 in V^* as $0(v) := 0$ for all $v \in V$. 0 satisfies:

$$(v_i^* + 0)(v) = (0 + v_i^*)(v) = v_i^*(v) \text{ for all } v \in V.$$

(d) (Existence of additive inverses)
For each element $v_i^* \in V^*$, we define an element in V^*, denoted by $-v_i^*$, with the properties that

$$(-v_i^*)(v) := -v_i^*(v).$$

Then it satisfies that

$$\left(v_i^* + (-v_i^*)\right)(v) = \left((-v_i^*) + v_i^*\right)(v) = 0 \quad \text{for all } v_i^* \in V^* \text{ and for all } v \in V.$$

(e) (Properties of scalar multiplication)
For all scalars α and β, we have

$$\begin{aligned}
\alpha(v_i^* + v_j^*)(v) &= \alpha v_i^*(v) + \alpha v_j^*(v) = \left(\alpha v_i^* + \alpha v_j^*\right)(v), \\
\left((\alpha + \beta)v_i^*\right)(v) &= \alpha v_i^*(v) + \beta v_i^*(v) = \left(\alpha v_i^* + \beta v_i^*\right)(v), \\
(\alpha\beta)v_i^*(v) &= \alpha\left(\beta v_i^*\right)(v), \\
1 v_i^*(v) &= v_i^*(v) \quad \text{for all } v \in V.
\end{aligned}$$

2. (inspired by [2]) Let $\{e_1, \ldots, e_n\}$ be a basis for V. Then every v in V is uniquely a linear combination $v = \sum \alpha_i e_i$ with $\alpha_i \in \mathbb{R}$. Let v_i^* in V^* be a linear map which picks out the i-th coordinate from $v \in V$, that is, $v_i^*(v) = \alpha_i$ for $v = \sum \alpha_i e_i$. v_i^* is characterized by

$$v_i^*(e_j) = \delta_j^i = \begin{cases} 1 \text{ (if } i = j), \\ 0 \text{ (if } i \neq j). \end{cases}$$

We prove that the maps $v_1^*, \ldots, v_n^*$ form a basis for V^*.
Firstly, we show that $v_1^*, \ldots, v_n^*$ span V^*. Indeed, if $v^* \in V^*$ and $v = \sum \alpha_i e_i$ in V, then

$$v^*(v) = \sum \alpha_i v^*(e_i) = \sum v^*(e_i) v_i^*(v).$$

Hence,

$$v^* = \sum v^*(e_i) v_i^*,$$

which shows that $v_1^*, \ldots, v_n^*$ span V^*.
To show linear independence, suppose $\sum c_i v_i^* = 0$ for some $c_i \in \mathbb{R}$. Applying both sides to the vector e_j gives

$$0 = \sum c_i v_i^*(e_j) = \sum c_i \delta_j^i = c_j, \quad j = 1, \ldots, n.$$

Hence, $v_1^*, \ldots, v_n^*$ are linearly independent. Thus,

$$\dim V^* = \dim V = n.$$

□

References

[1] Serge Richard. Lecture notes of Special Mathematics Lecture, Differential geometry (Fall 2024). 2024.

[2] Loring W Tu. An Introduction to Manifolds. Springer, 2011.