SML Fall 2024, Exercise 2.1.3.

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This answer is inspired by [1, Thm. IV.1.2] and the answer by Mr. Firdaus.

1 Reminder

- Let $C^\infty(p)$ be a set of all $\mathbb{R}-$valued smooth functions on $p \in M$. $C^\infty(p)$ is $\mathbb{R}-$algebra. That is,
  1. $C^\infty(p)$ is a vector space over $\mathbb{R}$ under addition and scalar multiplication,
  2. $C^\infty(p)$ is a ring under addition and multiplication,
  3. If $\alpha \in \mathbb{R}$ and $f, g \in C^\infty(p)$, then $\alpha(fg) = (\alpha f)g = f(\alpha g)$,

  where a multiplication in $C^\infty(p)$ is defined as $(fg)(p) := f(p)g(p)$ for $f, g \in C^\infty(p)$.
- (Definition 1.3.4., p.9) Let M and N be smooth manifolds. A map $F : M \to N$ is a smooth map if for any $p \in M$ there exist coordinate neighbourhoods (U, φ) of p and (V, ψ) of $F(p)$ with $F(U) \subset V$ such that the map

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

  is smooth.
- For any smooth map $F : M \to N$ between smooth manifolds M, N and for any $p \in M$, one sets the following two maps

$$F^* : C^\infty(F(p)) \to C^\infty(p) \text{ with } F^*(f) := f \circ F \qquad \forall f \in C^\infty(F(p)),$$
$$F_* : T_p(M) \to T_{F(p)}(N) \text{ with } [F_*(X_p)](f) = X_p(F^*(f)) \qquad \forall X_p \in T_p(M) \text{ and } \forall f \in C^\infty(F(p)).$$

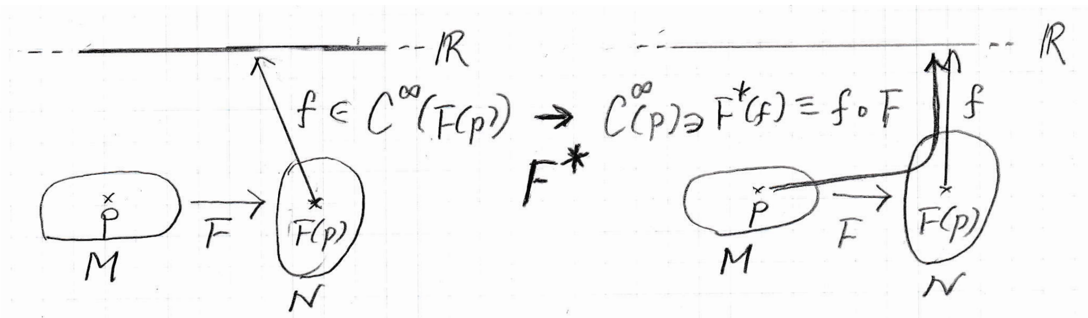

FIG. 1.1: An image representing a map F^*.

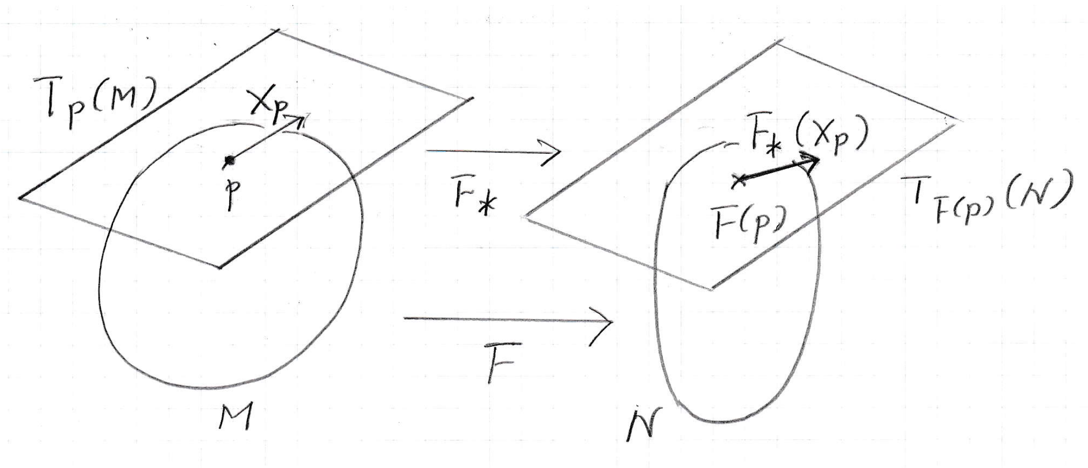

FIG. 1.2: An image representing a map F_*.

- A homomorphism is a map which preserves the structure of the domain.

2 Theorem and proof

Theorem 2.1 (Theorem 2.1.2. p.16)**.** In the framework introduced above, the map F^* is a homomorphism of algebras while the map F_* is a homomorphism of vector spaces. In addition, if $H = G \circ F$ is the composition of two smooth maps between manifolds, then $H^* = F^* \circ G^*$ and $H_* = G_* \circ F_*$.

Proof. 1. (The map F^* is a homomorphism of algebras.): $C^\infty(p)$ for $p \in M$ is an algebra on M, and $C^\infty(F(p))$ is an algebra on N. For a function $f \in C^\infty(F(p))$, $F^*(f) \equiv f \circ F$ is a smooth function defined on $p \in M$. Thus, F^* is a map from $C^\infty(F(p))$ to $C^\infty(p)$. F^* preserves the structure of $\mathbb{R}-$algebra of $C^\infty(F(p))$. Indeed, for any $f, g \in C^\infty(F(p))$ and for any $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} F^*(\alpha f + \beta g) &\equiv (\alpha f + \beta g) \circ F \\ &= \alpha f \circ F + \beta g \circ F \\ &= \alpha F^*(f) + \beta F^*(g), \end{aligned}$$

and

$$\begin{aligned} F^*(\alpha f g) &\equiv (\alpha f g) \circ F \\ &= \alpha (f \circ F)(g \circ F) \\ &= \alpha F^*(f) F^*(g) \end{aligned}$$

hold.

2. (The map F_* is a homomorphism of vector spaces.):

   (a) Firstly, we show that $F_*(X_p) : C^\infty(F(p)) \to \mathbb{R}$ is a tangent vector in $T_{F(p)}(N)$ for any tangent vector X_p in $T_p(M)$. Indeed, for any $\alpha, \beta \in \mathbb{R}$ and for any $f, g \in C^\infty(F(p))$,

$$\text{(Linearity)} \quad \begin{aligned} F_*(X_p)(\alpha f + \beta g) &= X_p\left(F^*(\alpha f + \beta g)\right) \\ &= \alpha X_p\left(F^*(f)\right) + \beta X_p\left(F^*(g)\right) \\ &= \alpha F_*(X_p)(f) + \beta F_*(X_p)(g), \end{aligned}$$

and

$$\text{(Leibniz rule)} \quad \begin{aligned} F_*(X_p)(fg) &= X_p\left(F^*(fg)\right) \\ &= X_p[(fg) \circ F] \\ &= X_p[(f \circ F)(g \circ F)] \\ &= X_p(f \circ F)\, g\left(F(p)\right) + f(F(p))\, X_p(g \circ F) \\ &= X_p\left(F^*(f)\right)\ g\left(F(p)\right) + f\left(F(p)\right)\ X_p\left(F^*(g)\right) \\ &= g\left(F(p)\right)\ F_*(X_p)(f) + f\left(F(p)\right)\ F_*(X_p)(g) \end{aligned}$$

   hold.

(b) Next, we show that F_* preserves linearity of a tangent vector space. Indeed,

$$
\begin{aligned}
[F_*(\alpha X_p + \beta Y_p)](f) &\equiv (\alpha X_p + \beta Y_p)\,(F^*(f)) \\
&= \alpha X_p(f \circ F) + \beta Y_p(f \circ F) \\
&= \alpha F_*(X_p)(f) + \beta F_*(Y_p)(f) \\
&= [\alpha F_*(X_p) + \beta F_*(Y_p)](f)
\end{aligned}
$$

holds.

From (a) and (b), F_* is proved to be a homomorphism of tangent vector spaces, $T_p(M)$ and $T_{F(p)}(N)$.

3. (For $H = G \circ F$, $H^* = F^* \circ G^*$ holds.): For any f in $C^\infty(H(p))$ $\forall p \in M$,

$$
\begin{aligned}
H^*(f) &\equiv f \circ H \\
&= f \circ (G \circ F) \\
&= (f \circ G) \circ F \\
&= G^*(f) \circ F \\
&= F^*\,(G^*(f)) \\
&= [F^* \circ G^*](f).
\end{aligned}
$$

4. (For $H = G \circ F$, $H_* = G_* \circ F_*$ holds.): For any X_p in $T_p(M)$ and f in $C^\infty(H(p))$ $\forall p \in M$,

$$
\begin{aligned}
[H_*(X_p)](f) &\equiv X_p\,(H^*(f)) \\
&= X_p\big(F^*\,(G^*(f))\big) \\
&= [F_*(X_p)]\,(G^*(f)) \\
&= G_*[F_*(X_p)](f) \\
&= [G_* \circ F_*(X_p)](f).
\end{aligned}
$$

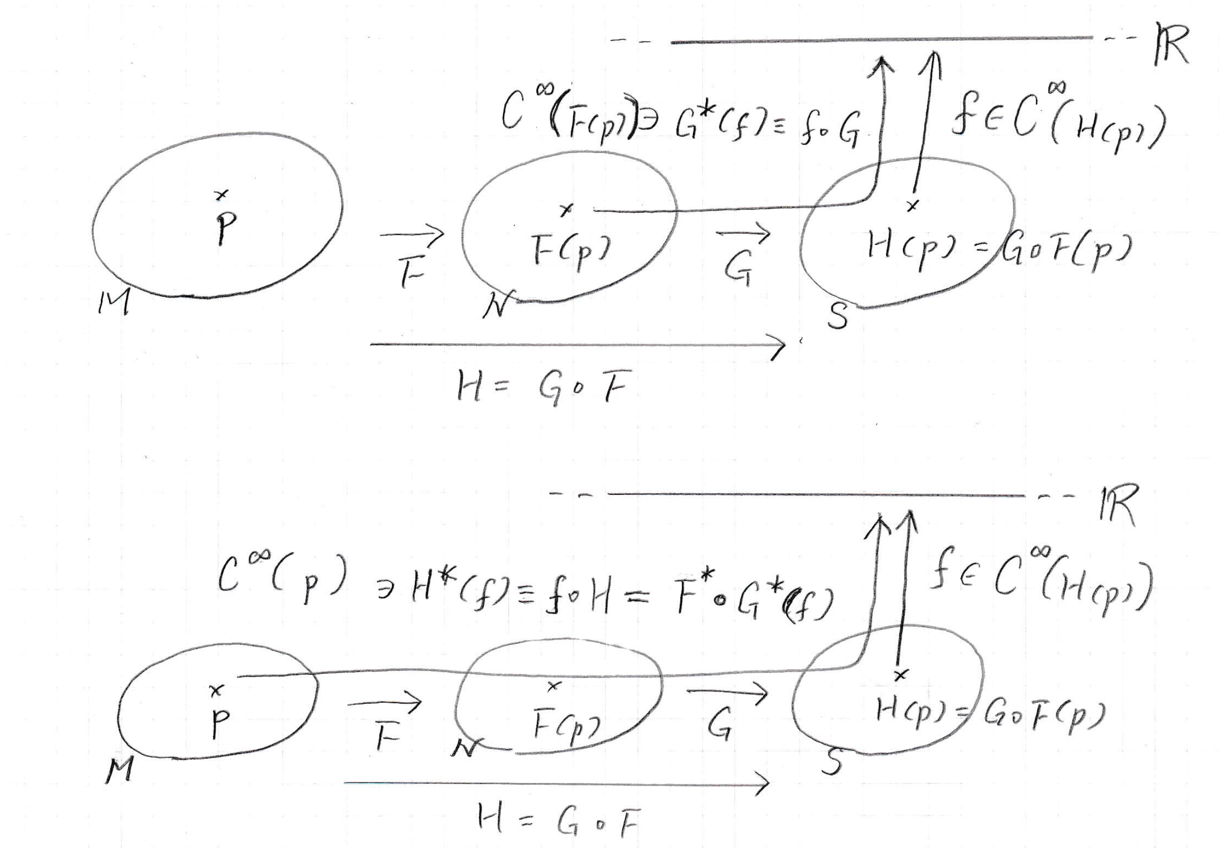

FIG. 2.1: G^* is a pull back of $f \in C^\infty(H(p))$ to $G^*(f) \in C^\infty(F(p))$,
$H^* = F^* \circ G^*$ is a pull back of $f \in C^\infty(H(p))$ to $H^*(f) \in C^\infty(p)$.

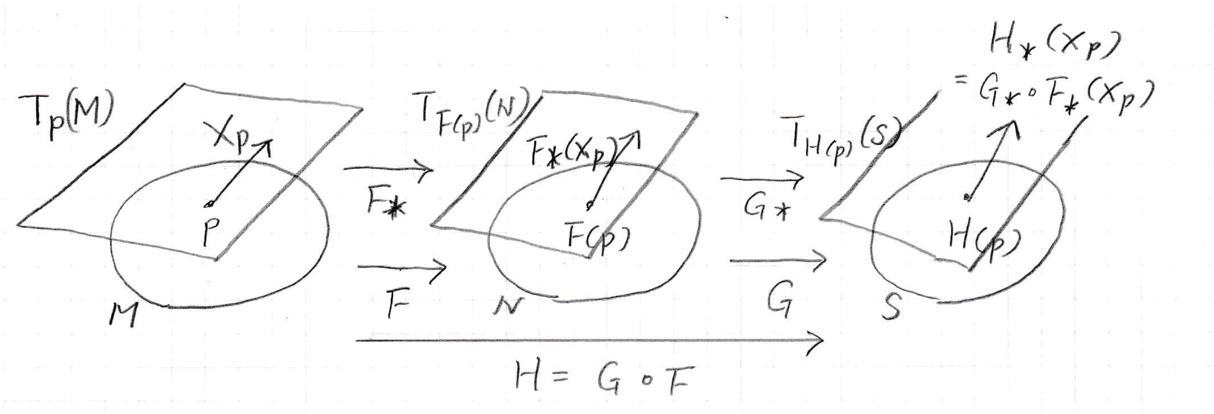

FIG. 2.2: F_* is a push forward of $X_p \in T_p(M)$ to $F_*(X_p) \in T_{F(p)}(N)$,
H$_*$ $= G_* \circ F_*$ is a push forward of $X_p \in T_p(M)$ to $H_*(X_p) \in T_{H(p)}(S)$.

□