Report 2 - Differential Geometry

1 Exercise 1.1.6

Study the separation axioms and exhibit some spaces which are not Hausdorff. Show that $\mathbb{R}^n$ is Hausdorff.

Answer :
An example of a non Hausdorff set is the following :

If M $= \{1,2,3\}$ and $\tau = \{\emptyset, \{1\}, \{2\}, \{1,2\}\{1,2,3\}\}$.

If we try to apply the definition of an Hausdorff space to (M, τ) and we take $p_1 = 1$ and $p_2 = 3$ then the only neighborhood of p_2 is $\{1,2,3\}$. We can choose $\{1\}$ or $\{1,2\}$ for p_1 but then $\{1,2,3\} \cap \{1\} = \{1\} \neq \emptyset$ and $\{1,2,3\} \cap \{1,2\} = \{1\} \neq \emptyset$. The space is not Hausdorff.

Be (M,τ) a topological space. Let's generalize the case of an element, p_1 which has only one neighborhood V_1 in which it is not the only element of M. Let's take p_2 in this neighborhood.

Case 1) p_2 has no other neighborhood. Then, both elements have the same unique neighborhood and the topological space is not Hausdorff.

Case 2) p_2 has at least another neighborhood V_2. Then $V_1 \cap V_2 = \{p_2\}$ and the topological space is not Hausdorff.

Then, having an element with only one neighborhood with more than one element (M in the case of topological space) unable you from being Hausdorff.

If we take the previous example back : M $= \{1,2,3\}$ and $\tau = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$ and try to make it Hausdorff we need to add an element in τ that would contain 3 and not 1. Such an element can either be $\{3\}$ or $\{2,3\}$.

If we add $\{3\}$, for (M, τ) to remain a topology, we have to add $\{3\} \cup \{2\} = \{3,2\}$ and $\{3\} \cup \{1\} = \{3,1\}$. τ is then the set of all possible subsets of M.

If we add only $\{2,3\}$ to τ we still have a topological space but:

. for 1 and 2 we can take $\{1\}$ and $\{2\}$;

. for 1 and 3 we can take $\{1\}$ and $\{2,3\}$;

. for 3 and 2 we have no solution;

Then, we need to add $\{3\}$ or $\{1,3\}$ to τ. As seen previously adding $\{3\}$ means adding $\{1,3\}$. Now, adding $\{1,3\}$ means adding $\{3\}$ since $\{1,3\} \cap \{3,2\} = \{3\}$.

Then, the only way to turn the previous example into an Hausdorff space is to take τ as the set of all possible subsets of M.

Now let's prove that $\mathbb{R}^n$ with the usual topology is Hausdorff. Let's take p_1 and p_2 in $\mathbb{R}^n$ such that $p_1 \neq p_2$. If $\mathbb{R}^n$ is Hausdorff then we can find V_1 and V_2 such that $V_1 \cap V_2 = \emptyset$.

Knowing the property of separability of the euclidean distance on $\mathbb{R}^n$, we know that d(p_1, p_2) $\neq 0$. Let's note this euclidean distance d. Since the usual topology on $\mathbb{R}^n$ is taking all open sets as τ, and since all open ball B(r, a) is an open set, let's show that p_1 and p_2 have neighborhoods and that we can find some that are disjointed.

For V_1 let's take the open ball of center p_1 and of radius $\frac{d}{2}$ and for V_2 the open ball of center p_2 and of radius $\frac{d}{2}$. If we recall the definition on an open ball : $V_1 = \mathrm{B}(p_1, \frac{d}{2}) = \{x \in \mathbb{R}^n : |x - p_1| < \frac{d}{2}\}$ and $V_2 = \mathrm{B}(p_2, \frac{d}{2}) = \{x \in \mathbb{R}^n : |x - p_2| < \frac{d}{2}\}$ and we have $V_1 \cap V_2 = \emptyset$.

Then, for every p_1, p_2 in $\mathbb{R}^n$ we can find V_1, V_2 such that $V_1 \cap V_2 = \emptyset$. Therefore, $\mathbb{R}^n$ is Hausdorff.

2 Exercise 1.1.13

Check that (U,τ_U) defines a topological space, and show that open sets of the subspace topology might not be open sets of the original topology

Answer :
Let's check that the three properties of a topological space are respected :

i) U and $\emptyset$ belong to τ_U :

M belongs to τ so $M \cap U$ belongs to τ_U. Furthermore, $U \subset M$ then $M \cap U = U$.

$\emptyset$ belongs to τ so $\emptyset \cap U = \emptyset$ belongs to τ_U.

The first property is respected.

ii) If $V_\alpha \in \tau_U$, then $\cup_\alpha V_\alpha \in \tau_U$, ($\tau_U$ stable for arbitrary unions)

Let $\cup_\alpha W_\alpha$ be an arbitrary union over τ_U. For all element W_α in τ_U, there exists an element V_α in τ such that $W_\alpha = V_\alpha \cap U$. Thus, $\cup_\alpha W_\alpha = \cup_\alpha (V_\alpha \cap U) = (\cup_\alpha V_\alpha) \cap U$ by distributivity of the union and intersection.

Since (M,τ) is a topological space, $\cup_\alpha V_\alpha$ is in τ and $(\cup_\alpha V_\alpha) \cap U$ is in τ_U.

The second property is respected.

iii) If $V_i \in \tau$, then $\cap_{i=1}^{N} Vi \in \tau$, ($\tau$ stable for finite intersections).

Let $\cap_{i=1}^{N} W_i$ be a finite intersection over τ_U. For all element W_i in τ_U, there exists an element V_i in τ such that $W_i = V_i \cap U$. Thus, $\cap_{i=1}^{N} W_i = \cap_{i=1}^{N} (V_i \cap U) = (\cap_{i=1}^{N} V_i) \cap U$ by distributivity of the union and intersection.

Since (M,τ) is a topological space, $\cap_{i=1}^{N} V_i$ is in τ and $(\cap_{i=1}^{N} V_i) \cap U$ is in τ_U.

Let's take a concrete example to show that element of the subspace topology might not be open sets of the original topology : M $= \{1, 2, 3\}$ and $\{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$.

If we take $U = \{3, 2\}$ then the subspace topology is (U, $\tau = \{\emptyset, \{2\}, \{2, 3\}\}$ with $\{2, 3\}$ which is not an open set of the original topology.